MATH 401 Applications of Linear Algebra

Problem Set on SVD and Pseudo-Inverse

Use the program `pix.m` which you can find on my webpage. You have three data choices. You must also enter the number of subdivisions in the mesh. Then you can approximate the original image by adding up to N terms of the singular value expansion of the image matrix.

1. Enter 10 for the mesh size. Then choose data choice 1. You will see a random 10×10 pattern of gray, white and black squares. This is the original picture. Now you can approximate this picture by adding up the terms of the SVD expansion of the matrix. How many terms in the expansion are necessary to get a recognizable approximation? Note how fast the singular values are decreasing.

2. Enter 40 for the mesh size. Then choose data choice 2. Now how many terms are necessary to get a good approximation of the original image? Compare the rate of decrease of the singular values of this image with the rate of decrease in part a). If you use only 4 terms in the expansion, how much do you save in storage space as compared to the storge space for the full 40×40 matrix?

3. Enter 100 for the mesh size. Choose data choice 3. This is as close as I can come to a fingerprint. Again note how many terms are necessary to get a good approximation. Relate this observation to the rate at which the singular values decrease. Compare with the rates of parts a) and b). How much storage space do you save if you use 20 terms in the expansion as compared to the full 100×100 matrix?

4. a) Construct a 7×7 matrix `myZ` that is all zeros except for a row of ones in row 3 and a column of ones in column 5. The image is a cross. From an inspection of the image, how many distinct rows are there? What does this tell you about the rank of your matrix Z? How many nonzero singular values do you expect this matrix to have? Run `pix` with data choice 4.

b) Finally, construct a 7×7 matrix `myZ` that is ones down the main diagonal and zeros elsewhere. What features of the diagonal matrix are different from the cross? Now how many distinct rows are there? What is the rank of this matrix Z? How many nonzero singular values are there?

Do the same for a matrix that is all zeros except for ones on the "cross diagonal" $(i + j = 8)$.

c) Now make an X. The matrix is all zeros except for ones down the main diagonal, $(i = j)$ and the "cross diagonal ", $(i+j = 8)$. Run `pix`. How many nonzero singular values are there? Why does the top-bottom symmetry (or the left-right symmetry) reduce the number of nonzero singular values by a factor of 2 (roughly)?

d) In general, if Z is symmetric about its middle row, or about its middle column, why is the number of nonsingular values less than or equal to $(n + 1)/2$?

The MATLAB command for the svd is `[U,S,V] = svd(A)`. S is the matrix Σ. The command to find the pseudo-inverse is `Aplus = pinv(A)`.

5. Let A be the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 2 & 3 \\ -1 & 1 & 2 & 2 \end{bmatrix}.$$

a) What is the rank of$A$? What is the null space of A?

b) Use MATLAB to find the svd of A.

c) Use MATLAB to find the pseudo inverse A^+.

d) Find the solution x^+ of the equation $Ax = b$ where $b = [1, 1, 1]$.

6. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 4 \\ 2 & 1 & 3 \\ -1 & 0 & -.9999 \\ 4 & -2 & 2 \end{bmatrix}.$$

a) Use MATLAB to find the svd of A. Notice that one of the singular values is quite small. What is the rank of A?

b) Use MATLAB to find the pseudo inverse A^+.

c) Find the solution x^+ of $Ax = b$ where $b = [1, 1, 1, 1, 1]$. Can you explain why the solution is so large?

d) Now change $A(4, 3)$ to -1. Call the modified matrix B. What is the rank of B?

e) Find the solution y^+ of $By = b$ using the pseudo inverse B^+. How do you explain the dramatic difference between x^+ and y^+ ?

f) What is the Frobenius norm $\|A - B\|_F$? This pair of matrices shows that a lower rank matrix B can be a very good approximation of A, but A^+ and B^+ can be very different.