Parametric Resonance in Wave Equations with a Time-Periodic Potential

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Abstract

We consider the wave equation in three space dimensions perturbed by a time-periodic potential with compact support in space, multiplied by a small parameter, ε. When $\varepsilon = 0$, the scattering theory of Lax and Phillips defines scattering frequencies which describe the decay of solutions in the neighborhood of the support of the potential. For $\varepsilon > 0$, scattering frequencies are defined; they are analogous to Floquet exponents. We show that when the frequency of the time-periodic potential is a multiple of the real part of a scattering frequency σ_0 for the time-independent case, resonance occurs. When ε increases from zero, the scattering frequency σ_0 splits in a symmetric fashion, defining outgoing solutions which decay faster or slower than those of the time-independent problem. An example is given in the case of spherical symmetry of the potential.

AMS Classifications: 34, 35, 46

Key words: Wave Equation, Scattering, Periodic, Resonance

1 Introduction

Consider the wave equation with a time-dependent potential

$$u_{tt} - \Delta u + q_0(x)u + \varepsilon p(t) q_1(x) u = 0. \tag{1.1}$$

Here $p(t)$ has period T and ε is a small parameter. We think of this equation as a PDE generalization of Hill's equation

$$u''(t) + q_0 u + \varepsilon p(t) u = 0. \tag{1.2}$$

If the period T of p is a suitable multiple of $2\pi/\sqrt{q_0}$, a resonance occurs, producing an exponentially growing solution of (1.2) (see [9]).

We look for similar behavior for the solutions of (1.1). Our study is motivated by our interest in the behavior of scattering frequencies for (1.1) when q_0 and q_1 have compact support in R^3.

In section 2 we recall some elements of Kato's treatment of analytic perturbations of linear operators [4]. In section 3 we apply this theory to an abstract evolution equation in a complex Hilbert space H,

$$u_t = Au + \varepsilon p(t) Q u. \tag{1.3}$$

We assume that A generates a strongly continuous semigroup of contraction operators $U(t)$ on H and that Q is a bounded operator on H. We assume that A and Q take real vectors into real vectors. For certain values of T, $U(T)$ will have a real eigenvector λ_0. We assume that λ_0 is an isolated point of the spectrum of $U(T)$ with finite multiplicity and no generalized eigenvalues. We show that when p has period T, the eigenvalue λ_0 splits in a symmetric fashion into several branches $\lambda_j(\varepsilon)$ determined by the eigenvectors of A.

In section 4, we apply the results of section 3 to (1.1) defined for $(x,t) \in R^3 \times R$ where $q_0(x)$ and $q_1(x)$ have compact support. We do not apply the theory directly to the solutions of (1.1) but rather to a local semigroup $Z(t)$ associated with the solutions of (1.1) with $\varepsilon = 0$. This local semigroup (discussed by Lax and Phillips in [6]) describes the behavior of the solutions in a neighborhood of the support of the scattering potential. In particular $Z(t)$ is compact, and has eigenvalues $\exp(i\sigma t)$. The complex numbers σ are called the *scattering frequencies* for (1.1) with $\varepsilon = 0$. The finite energy solutions of (1.1) with $\varepsilon = 0$ decay exponentially in the local energy norm which corresponds to the fact that the scattering frequencies σ satisfy $Im(\sigma) > 0$. Scattering frequencies are also defined for (1.1) when $\varepsilon \neq 0$ (see [1]). In [2], it was shown that the scattering frequencies of (1.1) depend on

ε in a continuous fashion. Numerical computations of the scattering frequencies for equations like (1.1) were done in [7] and [8]. These computations showed that the effect of the periodic perturbation may be to force some of the scattering frequencies to the lower half plane, corresponding to exponentially growing solutions of (1.1). In this paper we show that if $\sigma_0 = \nu_0 + i\kappa_0$ is a scattering frequency for (1.1) with $\varepsilon = 0$, with no generalized eigenvectors, and $T = 2\pi/\nu_0$, then σ_0 splits into several branches when $\varepsilon \neq 0$. The directions of the splitting are symmetric with respect to the origin, and with respect to the imaginary axis. An example of this splitting is given for spherically symmetric solutions when q_0 and q_1 are real constants. In this case the resonant scattering frequency splits along a vertical line with one branch heading south and one branch heading north. At this time we are still unable to show that as ε increases, one of the resonant scattering frequencies crosses the real axis.

Finally we remark that there is a large literature which treats the Schrödinger equation with a time-periodic potential (for a survey, see the article of Howland [3]). This approach, which uses a quasi-enegry, did not seem to yield any additional results because of the special nature of the semigroup $Z(t)$. Furthermore our results do not seem to apply to the Schrödinger case because of our hypothesis that the eigenvalue λ_0 be an isolated point of the spectrum of $U(T)$.

2 Analytic perturbation theory

In this section we recall several results from the theory of analytic perturbations of the spectrum of a bounded operator. The standard reference is Kato [4].

Let $\varepsilon \to L(\varepsilon)$ be a holomorphic family of bounded linear operators on a complex Hilbert space H, defined on a neighborhood of $\varepsilon = 0$. We abbreviate $L(0)$ by writing simply L. $L(\varepsilon)$ may be expanded in a power series, convergent in the operator norm,

$$L(\varepsilon) = L + \varepsilon L_1 + \varepsilon^2 L_2 + \dots . \tag{2.1}$$

$L_n = L^{(n)}(0)/n!$ are bounded operators on H.

Let $\lambda_0 \in C$ be an isolated point of the spectrum of L which is an eigenvalue of geometric and algebraic multiplicity m. In this case we can apply the finite dimensiononal theory. λ_0 is a semisimple eigenvalue in the terminology of Kato. The eigenvalue λ_0 may split into several branches $\{\lambda_1(\varepsilon), \dots, \lambda_s(\varepsilon)\}$, $1 \leq s \leq m$. Let $D = \{|\varepsilon| < \varepsilon_0\}$ and $D_0 = D - \{0\}$.

Theorem 1 : Each of the branches $\lambda_j(\varepsilon)$ is differentiable at $\varepsilon = 0$ and holomorphic on D_0 for ε_0 sufficiently small.

Let Γ be a small circle that encloses λ_0 and $\lambda_j(\varepsilon)$ for $|\varepsilon| \leq \varepsilon_0$, and such that Γ does not meet any other part of the spectrum of $L(\varepsilon)$. Let $R(\zeta, \varepsilon) = (L(\varepsilon) - \zeta)^{-1}$ be the resolvant. Then define

$$P(\varepsilon) = -\frac{1}{2\pi i} \int_\Gamma R(\zeta, \varepsilon) d\zeta. \tag{2.2}$$

$P(\varepsilon)$ is the projection on the total eigenspace corresponding to the eigenvalues $\{\lambda_1(\varepsilon), \dots, \lambda_s(\varepsilon)\}$.

Theorem 2: $\varepsilon \to P(\varepsilon)$ is holomorphic on D. The dimension of the range of $P(\varepsilon)$ is constant, equal to m.

Let E denote the (finite dimensional) eigenspace of λ_0 and let $P = P(0)$ denote the projection onto E. Note that the adjoint of P, P^*, is the projection onto the eigenspace E^* of L^* for the eigenvalue $\bar{\lambda}_0$. It is easy to verify that

$$P^* = -\frac{1}{2\pi i} \int_{\Gamma^*} R^*(\zeta, 0) d\zeta \tag{2.3}$$

where Γ^* is a small circle containing $\bar{\lambda}_0$.

Because λ_0 is a semisimple eigenvalue, the function $\varepsilon \to (L(\varepsilon) - \lambda_0)P(\varepsilon)$ is holomorphic on D and vanishes at $\varepsilon = 0$. Thus it may be expanded in a convergent power series

$$(L(\varepsilon) - \lambda_0)P(\varepsilon) = \sum_{1}^{\infty} \varepsilon^n \tilde{L}_n. \tag{2.4}$$

The operators $\tilde{L}_n$ are determined from the L_n as follows. First we expand $R(\zeta, \varepsilon)$ in a power series in ε:

$$R(\zeta, \varepsilon) = R(\zeta) + R_1(\zeta)\varepsilon + R_2(\zeta)\varepsilon^2 + \ldots \tag{2.5}$$

where $R(\zeta) = R(\zeta, 0)$ and

$$R_1(\zeta) = -R(\zeta)L_1R(\zeta) \tag{2.6}$$

$$R_2(\zeta) = -R(\zeta)L_2R(\zeta) + R(\zeta)L_1R(\zeta)L_1R(\zeta). \tag{2.7}$$

Now since

$$(L(\varepsilon) - \lambda_0)P(\varepsilon) = -\frac{1}{2\pi i}\int_\Gamma (\zeta - \lambda_0)R(\zeta, \varepsilon)d\zeta, \tag{2.8}$$

we may substitute the expansion (2.5) into (2.8) and use (2.6) and (2.7) to yield the expressions for $\tilde{L}_1$ and $\tilde{L}_2$,

$$\tilde{L}_1 = \frac{1}{2\pi i}\int_\Gamma (\zeta - \lambda_0)R(\zeta)L_1R(\zeta)d\zeta \tag{2.9}$$

$$\tilde{L}_2 = \frac{1}{2\pi i}\int_\Gamma (\zeta - \lambda_0)[R(\zeta)L_2R(\zeta) - R(\zeta)L_1R(\zeta)L_1R(\zeta)]d\zeta. \tag{2.10}$$

We can get more explicit expressions for $\tilde{L}_1$ and $\tilde{L}_2$ using the Laurent expansion for $R(\zeta)$ at λ_0:

$$R(\zeta) = -P(\zeta - \lambda_0)^{-1} + \sum_{1}^{\infty} S^n(\zeta - \lambda_0)^n \tag{2.11}$$

where S is the *reduced resolvant* of L. S is defined by $S = (L - \lambda_0)^{-1}(I - P)$ on the range of $I - P$, and $S = 0$ on E. Substitute (2.11) into (2.9) and (2.10) and perform the residue calculation. We obtain

$$\tilde{L}_1 = PL_1P, \tag{2.12}$$

$$\tilde{L}_2 = PL_2P - PL_1PL_1S - PL_1SL_1P - SL_1PL_1P \tag{2.13}$$

$$= PL_2P - \tilde{L}_1L_1S - SL_1\tilde{L}_1 - PL_1SL_1P.$$

Theorem 3: For each branch $\lambda_j(\varepsilon)$, $\lambda_j'(0)$ is an eigenvalue of $\tilde{L}_1$. Furthermore if $\varphi(\varepsilon)$ is a continuous family of eigenvectors of $L(\varepsilon)$ with eigenvalue $\lambda_j(\varepsilon)$,

$$L(\varepsilon)\varphi(\varepsilon) = \lambda_j(\varepsilon)\varphi(\varepsilon), \tag{2.14}$$

then $\varphi(0)$ is an eigenvector of $\tilde{L}_1$,

$$\tilde{L}_1\varphi(0) = \lambda_j'(0)\varphi(0). \tag{2.15}$$

Proof: For each ε, let $\varphi(\varepsilon)$ be an eigenvector of $L(\varepsilon)$ with eigenvalue $\lambda_j(\varepsilon)$, that is, (2.14). We may assume that $\|\varphi(\varepsilon)\| = 1$. Then because the unit ball in H is weakly compact, we can extract a subsequence $\varepsilon_k \to \infty$ such that $\varphi(\varepsilon_k) \to w$ weakly in H. Since $\|\varphi(\varepsilon_k)\| = 1$ we also have $\varphi(\varepsilon_k) \to w$ strongly in H. We rewrite the left side of (2.4) as:

$$(L(\varepsilon_k) - \lambda_0)P(\varepsilon_k)\varphi(\varepsilon_k) = (L(\varepsilon_k) - \lambda_j(\varepsilon_k) + \lambda_j(\varepsilon_k) - \lambda_0)\varphi(\varepsilon_k)$$

$$= (\lambda_j(\varepsilon_k) - \lambda_0)\varphi(\varepsilon_k).$$

Then dividing (2.4) by ε_k we have

$$\frac{\lambda(\varepsilon_k) - \lambda_0}{\varepsilon_k}\varphi(\varepsilon_k) = \tilde{L}_1\varphi(\varepsilon_k) + \sum_{n=2}^{\infty} \varepsilon_k^{n-1}\tilde{L}_n\varphi(\varepsilon_k).$$

We take the limit as $k \to \infty$ and deduce that $\lambda_j'(0)w = \tilde{L}_1 w$. Of course, the same results hold if $\varphi(\varepsilon)$ is any continuous family of eigenvectors satisfying (2.14).

If $\lambda_j'(0)$ is a semisimple eigenvalue of $\tilde{L}_1$, we may apply Theorem 1, and deduce that $\lambda_j(\varepsilon)$ is twice differentiable at $\varepsilon = 0$.

Theorem 4: Assume that $\varphi(\varepsilon)$ is a continuous family of eigenvectors satisfying (2.14). Assume that $\tilde{L}_1 = 0$. Then $\lambda_j'(0) = 0$ for each j and $\varphi(0)$ is an eigenvector of $\tilde{L}_2$ with eigenvalue $2\lambda_j''(0)$,

$$\tilde{L}_2\varphi(0) = 2\lambda_j''(0)\varphi(0). \tag{2.16}$$

Note that in this case, (2.13) becomes

$$\tilde{L}_2 = PL_2P - PL_1SL_1P. \tag{2.17}$$

Proof: Since $\tilde{L}_1 = 0$, (2.5) yields

$$\tilde{L}_2\varphi(\varepsilon) + \sum_{n=3}^{\infty} \varepsilon^{n-2}\tilde{L}_n\varphi(\varepsilon) = \frac{L(\varepsilon) - \lambda_0}{\varepsilon^2} P(\varepsilon)\varphi(\varepsilon)$$

$$= \frac{\lambda(\varepsilon) - \lambda_0}{\varepsilon^2}\varphi(\varepsilon).$$

Now taking the limit as $\varepsilon \to 0$ yields (2.16).

3 Parametric resonance

Let H be a complex Hilbert space, with scalar product (u, v) and norm $\|u\|$. Let A be the generator of a C_0 contraction semigroup $U(t)$ on H, $\|U(t)\| \leq 1$. Let Q be a bounded operator on H and let $p(t)$ be a continuous, real valued function. We assume

$$u \in D(A) \iff \bar{u} \in D(A), \quad \overline{Au} = A\bar{u}, \quad \overline{Qu} = Q\bar{u} \tag{3.1}$$

and

$$p \text{ has period } T \text{ with } \int_0^T p(t)dt = 0. \tag{3.2}$$

We consider the abstract differential equation for a function $t \to u(t)$ taking values in H,

$$\frac{du}{dt} = Au + \varepsilon p(t)Qu, \tag{3.3}$$

where ε is a small parameter.

We assume that $\mu = -\kappa_0 + i\nu_0$, with $\kappa_0 \geq 0$ and $\nu_0 > 0$, is an eigenvalue of A with geometric and algebraic multiplicity $m \geq 1$. Thus there are independent eigenvectors

$$A\varphi_j = \mu\varphi_j, \quad j = 1, \ldots, m. \tag{3.4}$$

Because of (3.1), $\bar{\mu}$ is also an eigenvalue for A, with eigenvectors $\bar{\varphi}_j$:

$$A\bar{\varphi}_j = \bar{\mu}\bar{\varphi}_j, \quad j = 1, \ldots, m.$$

Finally we assume

$$\nu_0 = 2\pi/T \tag{3.5}$$

and we assume that

$$\lambda_0 = e^{\mu T} = e^{-\kappa_0 T} \tag{3.6}$$

is an isolated point of the sprectrum of $U(T)$.

It follows that λ_0 is an eigenvalue of $U(T)$ of algebraic and geometric multiplicity $2m$. Let E denote the $2m$ dimensional eigenspace.

The solution of the initial value problem for (3.3) is given by an evolution operator $U_\varepsilon(t, s)$ for $s \leq t$, see Kato [5]. Integrating (3.3), we have

$$U_\varepsilon(t,s) = U(t-s) + \varepsilon \int_s^t U(t-\tau)p(\tau)QU_\varepsilon(\tau,s)d\tau \quad \text{for } s \le t. \tag{3.7}$$

Theorem 5: $\varepsilon \to U_\varepsilon(t,s)$ is an entire holomorphic family of bounded evolution operators $U_\varepsilon(t,s) : H \to H$ for $s \le t$.

Proof: We compute the derivatives of $U_\varepsilon(t,s)$ at $\varepsilon = 0$ formally, and show that these derivatives yield a convergent power series. For $f \in H$ and s fixed, let

$$z(t,\varepsilon) = U_\varepsilon(t,s)f, \quad t \ge s.$$

Then $z(t,\varepsilon)$ satisfies

$$z(t,\varepsilon) = U(t-s)f + \varepsilon \int_s^t U(t-\tau)p(\tau)Qz(\tau,\varepsilon)d\tau.$$

Formally differentiating, we see that

$$\frac{\partial z(t,\varepsilon)}{\partial \varepsilon} = \int_s^t U(t-\tau)p(\tau)Qz(\tau,\varepsilon)d\tau + \varepsilon \int_s^t U(t-\tau)p(\tau)Q\frac{\partial z(\tau,\varepsilon)}{\partial \varepsilon}d\tau, \tag{3.8}$$

so that

$$z_1(t) \equiv \frac{\partial z(t,0)}{\partial \varepsilon} = \int_s^t U(t-\tau)p(\tau)z_0(\tau)d\tau \tag{3.9}$$

where $z_0(\tau) = U(\tau)f$. In general we have

$$z_n(t) \equiv \frac{\partial^n z(t,0)}{\partial \varepsilon^n} = n \int_s^t U(t-\tau)p(\tau)Qz_{n-1}(\tau)d\tau. \tag{3.10}$$

Let $\alpha = \max |p(t)|$. Equation (3.10) yields the inequality

$$\|z_n(t)\| \le \alpha \int_s^t \|U(t-\tau)\|\|Q\|\|z_{n-1}(\tau)\|d\tau$$

$$\le n\alpha\|Q\| \int_s^t \|z_{n-1}(\tau)\|d\tau.$$

By induction we find that

$$\|z_n(t)\| \le (\alpha\|Q\|(t-s))^n \|f\|.$$

Thus $z(t,\varepsilon)$ has the convergent power series

$$z(t,\varepsilon) = \sum_0^\infty \frac{z_n(t)\varepsilon^n}{n!} \tag{3.11}$$

with

$$\|z(t,\varepsilon)\| \leq \sum_0^\infty \frac{(\alpha\|Q\|(t-s)|\varepsilon|)^n}{n!}\|f\| \leq e^{\alpha\|Q\|(t-s)|\varepsilon|}\|f\|.$$

The theorem is proved.

Now we wish to apply the results of section 2 to the holomorphic family of operators

$$\varepsilon \to L(\varepsilon) \equiv U_\varepsilon(T,0)$$

and investigate the behavior of the eigenvalues that split from λ_0 for $\varepsilon \neq 0$. From (3.9) and (3.10) we see that

$$L_1 f = \int_0^T U(T-s)p(s)QU(s)f\,ds, \tag{3.12}$$

and

$$L_2 f = \int_0^T U(T-s)p(s)QU(s)\int_0^s U(s-\tau)p(\tau)QU(\tau)f\,d\tau\,ds \tag{3.13}$$

To get more information about how the eigenvalue λ_0 splits, we use the $2m$ dimensional basis of eigenvectors of E, arranged as follows:

$$\varphi_1, \bar{\varphi}_1, \ldots, \varphi_m, \bar{\varphi}_m \tag{3.14}$$

We introduce a basis for the $2m$ dimensional eigenspace E^* of $U(T)^*$ (with the same real eigenvalue $\lambda_0 = \exp(-\kappa_0 T)$). Let $\psi_1, \ldots, \psi_m$ be the eigenvectors of A^* with eigenvalue $\bar{\mu}$:

$$A^*\psi_j = \bar{\mu}\psi_j, \quad j = 1, \ldots, m.$$

Then because of (3.1), $\bar{\psi}_j$ are also eigenvectors of A^*

$$A^*\bar{\psi}_j = \mu\bar{\psi}_j, \quad j = 1, \ldots, m.$$

We take the basis of E^* as $\psi_1, \bar{\psi}_1, \ldots, \psi_m, \bar{\psi}_m$. We observe that

$$(\varphi_j, \bar{\psi}_k) = 0, \quad 1 \leq j,k \leq m, \tag{3.15}$$

whence

$$(\bar{\varphi}_j, \psi_k) = \overline{(\varphi_j, \bar{\psi}_k)} = 0, \quad 1 \leq j,k \leq m.$$

This is easily seen since

$$\mu(\varphi_j, \bar{\psi}_k) = (A\varphi_j, \bar{\psi}_k) = (\varphi_j, A^*\bar{\psi}_k) = (\varphi_j, \mu\bar{\psi}_k) = \bar{\mu}(\varphi_j, \bar{\psi}_k).$$

Since we assume $\nu_0 = Im(\mu) > 0$, this implies (3.15).

Let N be the $2m \times 2m$ matrix consisting of the 2×2 blocks

$$N_{i,j} = \begin{bmatrix} (\varphi_i, \psi_j) & (\bar{\varphi}_i, \psi_j) \\ (\varphi_i, \bar{\psi}_j) & (\bar{\varphi}_i, \bar{\psi}_j) \end{bmatrix} \quad i, j = 1 \ldots, m. \tag{3.16}$$

We let M be the $2m \times 2m$ matrix consisting of the 2×2 blocks

$$M_{i,j} = \begin{bmatrix} (L_1\varphi_i, \psi_j) & (L_1\bar{\varphi}, \psi_j) \\ (L_1\varphi_i, \bar{\psi}_j) & (L_1\bar{\varphi}_i, \bar{\psi}_j) \end{bmatrix}, \quad i, j = 1, \ldots, m. \tag{3.17}$$

Here L_1 is given by (3.12).

Theorem 6: Assume (3.1) and (3.2). Then the values $\rho_j = \lambda_j'(0)$ of the derivatives of the branches $\lambda_j(\varepsilon)$ are the roots of the characteristic equation

$$\det(\rho N - M) = 0. \tag{3.18}$$

Furthermore, the roots ρ of (3.18) are symmetric with respect to the origin, and with respect to the real axis.

Proof: From Theorem 3, we know that the derivatives $\lambda_j'(0)$ are the eigenvalues of $\tilde{L}_1 = L_1 P L_1$. Let w be an eigenvector of $\tilde{L}_1$ with eigenvalue ρ,

$$\rho w = \tilde{L}_1 w = P L_1 w, \tag{3.19}$$

because $w \in E$. Now take scalar products with ψ_j and $\bar{\psi}_j$ in (3.19),

$$\rho(w, \psi_j) = (PL_1 w, \psi_j) = (L_1 w, P^* \psi_j) = (L_1 w, \psi_j) \tag{3.20}$$

and

$$\rho(w, \bar{\psi}_j) = (L_1 w, \bar{\psi}_j) \tag{3.21}$$

because $\psi_j, \bar{\psi}_j \in E^*$ and $P^* = I$ on E^*. Since $w \in E$, we can express w uniquely as

$$w = \sum_{j=1}^{m} a_{2j-1}\varphi_j + \sum_{j=1}^{m} a_{2j}\bar{\varphi}_j.$$

Substituting this expression into (3.20) and (3.21) yields

$$\rho N a = M a$$

where $a = (a_1, a_2, \ldots, a_{2m})$. This is equivalent to (3.18).

By (3.15), the blocks

$$N_{i,j} = \begin{bmatrix} n_{i,j} & 0 \\ 0 & \bar{n}_{i,j} \end{bmatrix}$$

where $n_{i,j} = (\varphi_i, \psi_j)$.

To compute the elements of $M_{i,j}$, we use (3.12).

$$(L_1\varphi_i, \psi_j) = \int_0^T (U(T-s)p(s)QU(s)\varphi_i, \psi_j)ds = \int_0^T p(s)(QU(s)\varphi_i, U(T-s)^*\psi_j)ds$$

$$= \int_0^T p(s)e^{\mu s}(Q\varphi_i, e^{\bar{\mu}(T-s)}\psi_j) = e^{\mu T}(Q\varphi_i, \psi_j)\int_0^T p(s)ds = 0$$

by (3.2). Because $U(t)$ and Q take real vectors into real vectors,

$$(L_1\bar{\varphi}_i, \bar{\psi}_j) = \overline{(L_1\varphi_i, \psi_j)} = 0.$$

Furthermore,

$$m_{i,j} \equiv (L_1\bar{\varphi}_i, \psi_j) = e^{\mu T}(Q\bar{\varphi}_i, \psi_j)\int_0^T p(s)e^{(\bar{\mu}-\mu)s}ds \tag{3.22}$$

$$= e^{-\kappa_0 T}(Q\bar{\varphi}_i, \psi_j)p_2$$

where

$$p_2 = \int_0^T p(t)e^{-2i\nu_0 t}dt.$$

Also $(L_1\varphi_i, \bar{\psi}_j) = \overline{(L_1\bar{\varphi}_i, \psi_j)}$. Thus the 2×2 blocks $M_{i,j}$ have the form

$$M_{i,j} = \begin{bmatrix} 0 & m_{i,j} \\ \bar{m}_{i,j} & 0 \end{bmatrix}.$$

The matrix $\rho N - M$ consists of the 2×2 blocks

$$\rho N_{i,j} - M_{i,j} = \begin{bmatrix} \rho n_{i,j} & -m_{i,j} \\ -\bar{m}_{i,j} & \rho\bar{n}_{i,j} \end{bmatrix}.$$

Let $l(\rho) = \det(\rho N - M)$, whence $\overline{l(\rho)} = \det(\bar{\rho}\bar{N} - \bar{M})$ with blocks

$$\bar{\rho}\bar{N}_{i,j} - \bar{M}_{i,j} = \begin{bmatrix} \bar{\rho}\bar{n}_{i,j} & -\bar{m}_{i,j} \\ -m_{i,j} & \bar{\rho}n_{i,j} \end{bmatrix}.$$

Now interchange rows $2i-1$ and $2i$, $i = 1, \ldots, m$ and columns $2j-1$ and $2j$, $j = 1, \ldots, m$ of $\bar{\rho}\bar{N} - \bar{M}$. These row and column interchanges yield $\bar{\rho}N - M$. Hence

$$\overline{l(\rho)} = \det(\bar{\rho}\bar{N} - \bar{M}) = \det(\bar{\rho}N - M) = l(\bar{\rho}).$$

Thus $l(\rho)$ has real coefficients so that the roots come in conjugate pairs.

Next we show that $l(-\rho) = l(\rho)$. In fact $l(-\rho) = \det(-\rho N - M)$ and $-\rho N - M$ has the 2×2 blocks

$$\begin{bmatrix} -\rho n_{i,j} & -m_{i,j} \\ -\bar{m}_{i,j} & -\rho \bar{n}_{i,j} \end{bmatrix}.$$

Multiply the odd numbered rows by -1 and the even numbered columns by -1. These row and column operations on $-\rho N - M$ yield $\rho N - M$. Hence

$$l(-\rho) = \det(-\rho N - M) = \det(\rho N - M) = l(\rho).$$

This completes the proof of Theorem 6.

Remark: Let $\lambda_j(\varepsilon)$ be a branch of the eigenvalues splitting from λ_0, with eigenvector $\varphi_j(\varepsilon)$. Then the first term in an asymptotic expansion of $\varphi_j(\varepsilon)$ is an eigenvector of $\lambda_j'(0)N - M$.

When $\tilde{L}_1 = 0$, we see in (2.13) that

$$\tilde{L}_2 = PL_2P - PL_1SL_1P$$

where S is the reduced resolvant. Let K be the $2m \times 2m$ matrix consisting of the 2×2 blocks

$$K_{i,j} = \begin{bmatrix} (L_2\varphi_i - L_1SL_1\varphi_i, \psi_j) & (L_2\bar{\varphi}_i - L_1SL_1\bar{\varphi}_i, \psi_j) \\ (L_2\varphi_i - L_1SL_1\varphi_i, \bar{\psi}_j) & (L_2\bar{\varphi}_i, -L_1SL_1\bar{\varphi}_i, \bar{\psi}_j) \end{bmatrix}, \quad i, j = 1, \ldots, m. \tag{3.23}$$

Theorem 7: Assume (3.1) and (3.2). Suppose that $p_2 = 0$ and that $\det(N) \neq 0$. Then $\tilde{L}_1 = 0$ so that $\lambda_j'(0) = 0$ for each of the branches $\lambda_j(\varepsilon)$. In this case, $\lambda_j(\varepsilon)$ is twice differentiable at $\varepsilon = 0$ and $\rho_j = 2\lambda_j''(0)$ are the roots of the characteristic equation

$$\det(\rho N - K) = 0. \tag{3.24}$$

The roots of (3.24) are symmetric with respect to the real axis.

Proof: Since N is assumed to be invertible, the matrix for $\tilde{L}_1 = PL_1P$ in the basis (3.14) for the eigenspace E is $N^{-1}M$. In fact, for $f \in H$, the projection can be expressed

$$Pf = \sum_{j=1}^{m} c_{2j-1}\varphi_j + \sum_{j=1}^{m} c_{2j}\bar{\varphi}_j$$

where

$$c = N^{-1}((f,\psi_1),(f,\bar{\psi}_1),\ldots,(f,\psi_m),(f,\bar{\psi}_m)).$$

Now taking $f = L_1 g$ with

$$g = \sum_{j=1}^{m} a_{2j-1}\varphi_j + \sum_{j=1}^{m} a_{2j}\bar{\varphi}_j$$

we see that

$$((f,\psi_1),(f,\bar{\psi}_1),\ldots,(f,\psi_m),(f,\bar{\psi}_m)) = Ma.$$

Thus the coordinates of $\tilde{L}_1 g = PL_1 g$ are related to the coordinates of g by $c = N^{-1}Ma$.

Now assuming $p_2 = 0$, we see by (3.22) that $M = 0$, whence $\tilde{L}_1 = 0$. Thus $\lambda_j'(0) = 0$ for all branches of $\lambda_j(\varepsilon)$. By Theorem 5, λ_j is twice differentiable at $\varepsilon = 0$, and for each branch, $\rho_j = 2\lambda_j''(0)$ is an eigenvalue of $\tilde{L}_2$. If w is an eigenvector of $\tilde{L}_2$ with eigenvalue ρ,

$$\rho w = \tilde{L}_2 w,$$

we take scalar product with ψ_j and $\bar{\psi}_j$ to obtain

$$\rho(w,\psi_j) = (\tilde{L}_2,\psi_j)$$

$$= (PL_2 w - PL_1 SL_1 w, \psi_j) = (L_2 w - L_1 SL_1 w, \psi_j)$$

and similary for $\bar{\psi}_j$. Then writing $w = \sum a_{2j-2}\varphi_j + a_{2j}\bar{\varphi}_j$, we deduce, as in the proof of Theorem 6, that $\rho Na = Ka$ where K is given by (3.23). Finally we note that because L and S take real vectors into real vectors, the blocks $K_{i,j}$ have the form

$$K_{i,j} = \begin{bmatrix} k_{i,j} & l_{i,j} \\ \bar{l}_{i,j} & \bar{k}_{i,j} \end{bmatrix}.$$

It follows easily that if ρ satisfies (3.24), then so does $\bar{\rho}$.

Remark: The proof of Theorems 6 and 7 actually only depends on equation (3.7). Once this formula is established, we do not need to know the differential equation solved by $U_\varepsilon(t,s)$.

4 Scattering frequencies

We apply the results of sections 2 and 3 to the wave equation in three space dimensions with a time-periodic potential. Let $q_0(x) \in L^\infty(R^3)$, $x \in R^3$, with $q_0(x) \geq 0$ and $q_0(x) = 0$ for $|x| > 1$. We assume that $q_0(x) > 0$ on some open subset. Next let $q_1(x) \in L^\infty(R^3)$ be real valued with $q_1(x) = 0$ for $|x| > 1$. We consider the wave equations

$$u_{tt} - \Delta u + q_0(x)u = 0 \tag{4.1}$$

and

$$u_{tt} - \Delta u + q_0(x)u + \varepsilon p(t)q_1(x)u = 0 \tag{4.2}$$

where $p(t)$ is real valued, continuous, and has period T with

$$\int_0^T p(t)dt = 0. \tag{4.3}$$

We write (4.1) and (4.2) as systems

$$u_t = Au \tag{4.4}$$

and

$$u_t = Au + \varepsilon p(t)Qu \tag{4.5}$$

where now u is a pair, $u = [u(x,t), u_t(x,t)]$. A is the matrix differential operator

$$A = \begin{bmatrix} 0 & 1 \\ \Delta - q_0 & 0 \end{bmatrix} \tag{4.6}$$

and

$$Q = \begin{bmatrix} 0 & 1 \\ q_1 & 0 \end{bmatrix}. \tag{4.7}$$

The finite energy space H for equations (4.4) and (4.5) is the space of pairs $f = [f_1, f_2]$ which is the closure of $C_0^\infty(R^3) \times L^2(R^3)$ in the energy norm

$$\|f\| = [\int_{R^3} [|\nabla f_1|^2 + q_0|f_1|^2]dx + \int_{R^3} |f_2|^2 dx]^{1/2}.$$

The scalar product on H is

$$(f,g) = \int_{R^3} [\nabla f_1 \cdot \nabla \bar{g}_1 + q_0 f_1 \bar{g}_1 + f_2 \bar{g}_2]dx.$$

It is well known that A generates a unitary group $U(t) : H \to H$. Applying Theorem 5, we see that the finite energy solutions of (4.5) are given by an evolution operator $V_\varepsilon(t,s) : H \to H$, and that $\varepsilon \to V_\varepsilon(t,s)$ is an entire function with values in the space of bounded operators on H.

We shall not apply the results of section 3 directly to equations (4.4) and (4.5). In fact, A has no eigenvalues. Instead we study the behavior of the solutions in a neighborhood of the support of the potential. In [6] Lax and Phillips developed a framework to treat this situation. We recall some elements of that framework. First we make a decomposition of the space of data H into subspaces which represent solutions which have left the region of the potential (outgoing), those which have not reached the region of the potential (incoming), and the remainder which interacts with the potential. The *outgoing subspace* is

$$D^+ = \{f \in H : U_0(t)f = 0 \text{ in } |x| \leq t+1, t > 0\}$$

and the *incoming subspace* is

$$D^- = \{f \in H : U_0(t)f = 0 \text{ in } |x| \leq |t|+1, t < 0\}.$$

Here $U_0(t)$ is the unitary group of the free wave equation (with $q_0 = 0$). We can decompose H as an orthogonal direct sum

$$H = D^+ \oplus K \oplus D^-. \tag{4.8}$$

Let P_+ be the orthogonal projection on the *orthogonal complement* of D^+, which is $D^- \oplus K$. P_- will denote the orthogonal projection onto $D^+ \oplus K$. Note that for $f \in D^\pm$, $f = 0$ in $|x| < 1$. Hence for any $f \in H$,

$$P_\pm f = f \quad \text{for } |x| < 1. \tag{4.9}$$

Next we introduce the localized semigroup and evolution operator. We define

$$Z(t) = P_+ U(t) P_-, \quad \text{for } t \geq 0. \tag{4.10}$$

Because $U(t)D^+ \subset D^+$ for $t \geq 0$, $Z(t)f = 0$ for $f \in D^\pm$. For $f \in K$, $Z(t)f \in K$. $Z(t)$ is in fact a contraction semigroup on K. We denote its generator by B, $Z(t) = \exp(tB)$.

Singularities of the solutions of (4.1) propagate with speed one and are unimpeded by the potential. Hence for $f \in H$, $U(t)f$ is smooth in $\{|x| < 1\}$ for $|t| > 2$. This implies that the spectrum of B is discrete, consisting of eigenvalues μ_j, with $Re(\mu_j) < 0$. Furthermore, any vertical strip in the complex plane contains at most a finite number of eigenvalues. The *scattering frequencies* for (4.1) are the complex numbers

$$\sigma_j = \nu_j + i\kappa_j$$

such that $i\sigma_j = \mu_j$. Note that $\kappa_j > 0$.

Because $Z(t)$ is compact, the eigenspace associated with each eigenvalue μ_j is is finite dimensional. Assuming there are no generalized eigenvectors, solutions of (4.1) with data of compact support have an asymptotic expansion

$$u(x,t) \approx \sum_j e^{i\sigma_j t} v_j(x)$$

which approximates u in the local energy norm. The v_j are outgoing scattering eigensolutions of

$$-\Delta v + q_0(x)v = \sigma^2 v.$$

They satisfy the outgoing Sommerfeld radiation condition. The pair $[v_j, \mu_j v_j]$ does not belong to H because v_j grows exponentially as $|x| \to \infty$. However, for $|x| < 1$, the pair $[v_j, \mu_j v_j]$ agrees with an eigenvector φ_j of B with eigenvalue μ_j and

$$Z(t)\varphi_j = e^{\mu_j t}\varphi_j. \tag{4.11}$$

Now we define the localized evolution operator. Let

$$Z_\varepsilon(t,s) = P_+ V_\varepsilon(t,s) P_-.$$

Lemma: $Z_\varepsilon(t,s)$ is a holomorphic family of evolution operators on K. Z_ε and Z satisfy the integral relation (3.7).

Proof: That $Z_\varepsilon(t,s)$ is an evolution operator on K was shown in [1]. The holomorphic property is immediate because $\varepsilon \to V_\varepsilon(t,s)$ is holomorphic.

We need only verify that Z_ε and Z satisfy the integral relation (3.7). However, $U(t)$ and $V_\varepsilon(t,s)$ do satisfy this relation:

$$V_\varepsilon(t,s) = U(t-s) + \varepsilon \int_0^t U(t-\tau)p(\tau)QV_\varepsilon(\tau,s)ds.$$

Apply the projection P_+ to each term in this equation. Using the definitions of Z and Z_ε we find that for $f \in K$,

$$Z_\varepsilon(t,s)f = Z(t-s)f + \varepsilon \int_s^t Z(t-\tau)p(\tau)QV_\varepsilon(\tau,s)f d\tau.$$

But $QV_\varepsilon(\tau,s)f = QP_+V_\varepsilon(\tau,s)f = QZ_\varepsilon(\tau,s)f$ because $P_+g = g$ for $|x| < 1$. The lemma is proved.

Finally after this lengthy preparation, we can apply the results of sections 2 and 3 to the semigroup $Z(t)$ and the evolution operator $Z_\varepsilon(t,s)$.

Let $\sigma_0 = \nu_0 + i\kappa_0$ be a scattering frequency of (4.1) with $\nu_0 > 0$. Thus $\mu_0 = i\sigma_0$ is an eigenvalue of B. $\lambda_0 = \exp(\mu_0 T) = \exp(i\sigma_0 T) = \exp(-\kappa_0 T) > 0$ is an eigenvalue of $Z(T)$.

Theorem 8: Assume that the eigenvalue μ_0 has geometric and algebraic multiplicity m. Choose $T = 2\pi/\nu_0$, and assume that p is real valued, continous, and satisfies (3.2). Then the scattering frequency σ_0 splits into s branches $\sigma_j(\varepsilon)$, $1 \le s \le m$. $\sigma_j(\varepsilon)$ is differentiable at $\varepsilon = 0$ and holomorphic in a punctured complex neighborhood of $\varepsilon = 0$.

Proof: The general theory of section 2 applies here with $L(\varepsilon) = Z_\varepsilon(T,0)$. We need to verify that $\lambda_0 = \exp(\mu_0 T)$ is an isolated point of the spectrum of $L = L(0) = Z(T)$. But this follows because, as noted before, each vertical strip in the complex plane contains only a finite number of eigenvalues μ of B. This means that in each annulus centered at zero, $Z(T)$ has only a finite number of eigenvalues.

If $\lambda_j(\varepsilon)$ is a branch of the eigenvalues splitting from λ_0, then $\lambda_j'(0)$ exists. Now

$$\sigma_j(\varepsilon) = \frac{\log(\lambda_j(\varepsilon))}{iT}$$

are the branches of the scattering frequencies splitting from σ_0. They are holomorphic in a punctured neighborhood of $\varepsilon = 0$, and differentiable at $\varepsilon = 0$ with

$$\sigma_j'(0) = \frac{\lambda_j'(0)}{iT\lambda_0} = \frac{e^{\kappa_0 T}\lambda_j'(0)}{iT} \tag{4.12}$$

This ends the proof of Theorem 8.

Corollary: Let $\sigma_j(\varepsilon)$ be the branches of the scattering frequency which split from resonant scattering frequency σ_0. The $\sigma_j'(0)$ have complex values α such that if α is such a direction, then so is $-\bar{\alpha}$ and so is $-\alpha$.

This is an immediate consequence of Theorem 6, the remark at the end of section 3, and (4.12).

Example: We specialize the discussion to the case where q_0 is a real constant, $q_0 > 0$ on $\{|x| < 1\}$, and $q_1(x) \equiv 1$ on $\{|x| < 1\}$. If σ_0 is a scattering frequency of (4.1) that corresponds to a spherically symmetric scattering eigenfunction, then the perturbed solution of (4.2) will also be spherically symmetric. Thus we restrict our attention to spherically symmetric solutions of (4.1) and (4.2). We make the change of dependent variable $z(r,t) = ru(r,t)$. z now satisfies

$$z_{tt} - z_{rr} = \begin{cases} -q_0 z & 0 < r < 1 \\ 0 & r > 1 \end{cases}$$

$$z(0,t) = 0$$

$$z(r,0) = f_1(r), \quad z_t(r,0) = f_2(r).$$

Now if $f_1 = f_2 = 0$ for $r > 1$, then $z = z(t-r)$ for $t \geq 0$ and $r > 1$. Thus in this case, the values of z are completely determined by the solution of the initial boundary value problem

$$z_{tt} - z_{rr} + q_0 z = 0, \qquad 0 < r < 1 \tag{4.13}$$

$$z(0,t) = 0, \quad z_t + z_r = 0 \text{ on } r = 1 \tag{4.14}$$

$$z(r,0) = f_1(r), \quad z_t(r,0) = f_2(r).$$

Because $P_\pm f = f$ for $|x| < 1$, and $Z(t)f = P_+U(t)P_-f$, we see that when $f = 0$ for $|x| > 1$, $Z(t)f = U(t)f$ on $\{|x| < 1\}$. Consequently, in the case of spherical symmetry, the solutions of the localized evolution equation

$$\frac{dv}{dt} = Bv$$

are exactly the solutions of (4.13), (4.14), with $v = [z, z_t]$. The eigenfunctions of B are the spatial factors of the solutions of (4.13), (4.14) of the form $z(r,t) = \exp(\mu t)w(r)$ where w satisfies

$$-w_{rr} + q_0 + \mu^2 = 0$$

$$w(0) = 0, \quad \mu w(1) + w_r(1) = 0.$$

It is convenient to seek the solutions of this problem in the form $w(r) = \sin(\gamma r)$. γ and μ must satisfy

$$\gamma^2 + \mu^2 + q_0 = 0, \quad \mu\sin(\gamma) + \gamma\cos(\gamma) = 0. \tag{4.15}$$

Writing μ in terms of a scattering frequency, $\mu = i\sigma_0$, we see that for large q_0, γ and σ_0 have the asymptotic approximations

$$\gamma(q_0) = n\pi(1 + i/\sqrt{(q_0)} - 1/q_0) + O(q_0^{-3/2}),$$

$$\sigma_0(q_0) = \sqrt{\gamma^2 + q_0}.$$

We have taken the roots γ and σ_0 in the first quadrant. There is another root $\bar{\gamma}$ and corresponding root $-\bar{\sigma}_0$, which corresponds to $\bar{\mu}$.

In this case, the eigenvalue $\mu = i\sigma_0$ is simple ($m = 1$). The eigenfunctions of B and B^* are

$$\varphi(r) = \begin{bmatrix} \sin(\gamma r)/r \\ i\sigma_0 \sin(\gamma r)/r \end{bmatrix} \quad \text{and} \quad \psi(r) = \begin{bmatrix} \sin(\bar{\gamma})/r \\ i\bar{\sigma}_0 \sin(\bar{\gamma} r)/r \end{bmatrix}, \ 0 < r < 1.$$

The matrices M and N are 2×2 and the characteristic equation $\det(\rho N - M) = 0$ has the roots

$$\rho = \pm \frac{|m_{1,1}|}{|n_{1,1}|}$$

Here

$$n_{1,1} = (\varphi, \psi), \quad \text{and } m_{1,1} = e^{-\kappa_0 T} p_2(Q\bar{\varphi}, \psi).$$

Thus the scattering frequency σ_0 splits into two branches with

$$\sigma'(0) = \pm i \frac{|p_2(Q(\bar{\varphi}, \psi)|}{T|(\varphi, \psi)|}.$$

Remark: Computations show that when $p_2 \neq 0$, the branches of the resonant scattering frequency move vertically on the line $Re(\sigma) = Re(\sigma_0)$ as $\varepsilon \uparrow$, and eventually one of the branches crosses the real axis. This means that for $\varepsilon > 0$ sufficiently large, there are outgoing eigensolutions of the perturbed problem (4.2) that grow exponentially as $t \to \infty$. We hope to prove this in the future.

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