Math 140, Jeffrey Adams
Test I, SOLUTIONS

Question 1.
(a) The limit is $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$. With $f(x)=x^3+1$, $a=1$, $f(a)=2$ this becomes $\lim_{x\to 1}\frac{(x^3+1)-2}{x-1}$. This simplifies to $\lim_{x\to 1}\frac{x^3-1}{x-1}$. Plugging in $x=1$ into both numerator and denominator gives 0, so you need to factor. By long division $x^3-1=(x-1)(x^2+x+1)$. Therefore the limit is $\lim_{x\to x}\frac{(x-1)(x^2+x+1)}{x-1}=\lim_{x\to 1}(x^2+x+1)=3$.
(b) The equation of the tangent line is $\frac{y-2}{x-1}=3$, or $y-2=3(x-1)$, or $y=3x-1$.

Question 2.
(a) $\lim_{x\to 0}\frac{\sin(2x)}{3x}=\lim_{x\to 0}\frac{\sin(2x)}{2x}\frac{2}{3}=\frac{2}{3}\lim_{x\to 0}\frac{\sin(2x)}{2x}$. The inner limit is 1 (by substitution: let $u=2x$), so the answer is $\frac{2}{3}$.
(b) The numerator equals $2(x-1)$ and the denominator $(x-1)(x+1)$. Cancelling gives $\lim_{x\to 1}\frac{2}{x+1}$. Since $\lim_{x\to 1}(x+1)=2\neq 0$, this equals $\frac{2}{2}=1$.
(c) Divide the top by x^3 and the bottom by x^2 to give $\lim_{x\to\infty}\frac{x^3(1-\frac{2}{x^3})}{x^2(1+\frac{2}{x}-\frac{1}{x^2})}$. Since all the terms with x in the denominator go to 0, this equals $\lim_{x\to\infty}\frac{x^3}{x^2}=\lim_{x\to\infty}x=\infty$.
(d) If x is bigger than 1 then $x-1$ is positive, so $|x-1|=x-1$. Therefore the limit from the right is $\lim_{x\to a^+}(x-1)(1-x)=\lim_{x\to 1^+}(-1)=-1$.

Question 3.
(a) $\lim_{x\to 2^-}f(x)=\lim_{x\to 2^-}x=2$. Since $f(2)=2$, this says $\lim_{x\to 2^-}f(x)=f(2)$, which says f is continuous from the left at 2. Since $\lim_{x\to 2^+}f(x)=\lim_{x\to 2^+}=(x^2-1)=3$, and this does not equal $f(2)$, f is *not* continuous from the right at 2. Since it is not continuous from both the left and right at 2, f is *not* continuous at 2.
(b) Since $f(x)=x$ is continuous for $0<x<1$, and is continuous from the right at 0, and continuous from the left at 1 (it is also continuous from the right at 1, although this is irrelevant), we conclude f is continuous on $[0,1]$.

Question 4. Let $f(x)=\cos(x)-x$. We want to know if $f(x)=0$ for some $x>0$. Clearly f is continuous for all x. Now $f(0)=1-0=1>0$, and $f(\pi/2)=0-\pi/2<0$. By the intermediate value theorem $f(a)=0$ for some $0<a<\pi/2$.

Question 5.
(a) The limit is $\lim_{t\to 1}\frac{(t^2+3t-1)-(1+3-1)}{t-1}$.
(b) The numerator is equal to $t^2+3t-4=(t-1)(t+4)$. The limit is therefore $\lim_{t\to 1}\frac{(t-1)(t+4)}{t-1}=\lim_{t\to 1}(t+4)=5$, and the velocity at $t=1$ is 5.