Math 140, Jeffrey Adams

Test I, February 12, 2010 SOLUTIONS

Question 1.

(a) The slope of the line is

$$\begin{aligned}\lim_{x\to 3}\frac{f(x)-f(3)}{x-3} &= \lim_{x\to 3}\frac{x^2-3x-0}{x-3}\\ &= \lim_{x\to 3}\frac{x(x-3)}{x-3}\\ &= \lim_{x\to 3} x = 3.\end{aligned}$$

(b) The equation of the line is $\frac{y-0}{x-3}=3$, or $y=3x-9$.

Question 2.

(a) If it weren't for the absolute value, you could write $\frac{x^2-9}{x-3}=\frac{(x+3)(x-3)}{x-3}=x+3$, and the limit would be 6. However if x is slightly greater than 3 $\frac{|x^2-9|}{x-3}$ is positive (and near 6), where if x is slightly less than 3 it is negative (and near -6). Therefore the limit does not exist.

Another way:

$$\begin{aligned}\lim_{x\to 3}\frac{|x^2-9|}{x-3} &= \lim_{x\to 3}\frac{|(x-3)(x+3)|}{x-3}\\ &= \lim_{x\to 3}\frac{|x-3||x+3|}{x-3}\\ &= \lim_{x\to 3}\frac{|x-3|}{x-3}|x+3|\end{aligned}$$

Now

$$\frac{|x-3|}{x-3}=\begin{cases}1 & x>3\\ -1 & x<3\end{cases}$$

Of course $\lim_{x\to 3}|x+3|=6$, so the left and right limits are ± 6, so the two-sided limit doesn't exist.

(b)

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(x)}{\sin(2x)} &= \lim_{x\to 0}\frac{\sin(x)}{1}\cdot\frac{x}{x}\cdot\frac{1}{\sin(2x)}\cdot\frac{2x}{2x}\\ &= \lim_{x\to 0}\frac{\sin(x)}{x}\cdot\cdot\frac{2x}{\sin(2x)}\cdot\frac{1}{2}\\ &= \frac{1}{2}\lim_{x\to 0}\frac{\sin(x)}{x}\lim_{x\to 0}\frac{2x}{\sin(2x)}\\ &= \frac{1}{2}\lim_{x\to 0}\frac{\sin(x)}{x}\lim_{u\to 0}\frac{u}{\sin(u)}\\ &= \frac{1}{2}\cdot 1\cdot 1=\frac{1}{2}\end{aligned}$$

(c) Since $\cos(0)=1$, the limit is $\lim_{x\to 1}\sin(\pi)=0$.

(d) $\lim_{x\to\frac{\pi}{2}^+}\tan(x) = \lim_{x\to\frac{\pi}{2}^+}\frac{\sin(x)}{\cos(x)}$. Since $\cos(\frac{\pi}{2}) = 0$ and $\sin(\frac{\pi}{2}) = 1$ this is $\pm\infty$. If x is slightly bigger than $\frac{\pi}{2}$ then $\cos(x) < 0$, so the answer is $-\infty$.

Question 3. (a) Since e^x is continuous, $\lim_{x\to 0} e^x = e^0 = 1$, so $\lim_{x\to 0^+} f(x) = 1$. On the other hand $\lim_{x\to 0} 1 = 1$, so $\lim_{x\to 0^-} f(x) = 1$.
(b) Since both limits are equal, and are equal to the value of the function at 0, the function is left continuous, right continuous, and continuous.

Question 4.
The possibilities for asymptotes are $x = -1$ or $x - 2$. The function is equal to

$$\begin{aligned}\frac{(x^2-4)(x^2-1)}{(x+1)(x-2)^2} &= \frac{(x-2)(x+2)(x-1)(x+1)}{(x+1)(x-2)^2}\\ &= \frac{(x+2)(x-1)}{(x-2)}\end{aligned}$$

and the only vertical asymptote is at $x = 2$.
Question 5. This is the intermediate value theorem. Since $\sin(0) = 0$ and $\sin(\frac{\pi}{2}) = 1$, and sin is continuous, by the IVP $\sin(x)$ takes all values between 0 and 1. In particular since $0 < \frac{1}{e} < 1$, there is a number x (between 0 and $\frac{\pi}{2}$ in fact) such that $\sin(x) = \frac{1}{e}$.