**Math 140, Jeffrey Adams**

Test III, November 6, 1998 SOLUTIONS

Question 1. (25 points)

(a) $f'(x) = 3x^2 - 12x + 9$. Setting this equal to zero gives $3(x^2 - 4x + 3) = 0$. This factors to give $3(x-1)(x-3) = 0$, so $x = 1, 3$. The only point in the region is 1. So the only local extreme point is $x = 1$. Check this point and the endpoints: $f(1) = 1 - 6 + 9 - 1 = 3$, $f(0) = -1$ and $f(2) = 8 - 24 + 18 - 1 = 1$. The maximum value is 3, at $x = 1$. The minimum value is $-1$, at $x = 0$.

(b) $\lim_{x\to\pm\infty} \frac{x^2+x+1}{x^3-x} = \lim_{x\to\pm\infty} \frac{x^2(\frac{1}{x}+\frac{1}{x^2})}{x^3(1-\frac{1}{x^2})} = \lim_{x\to\pm\infty} \frac{(\frac{1}{x}+\frac{1}{x^2})}{x(1-\frac{1}{x^2})}$ The terms in parentheses go to zero, to give $\lim_{x\to\pm\infty} \frac{1}{x} = 0$. So $y = 0$ is a horizontal asymptote to the right and left. The vertical asymptotes are found where the denominator equals 0. Since $x^3 - x = x(x^2-1) = x(x-1)(x+1)$, there are vertical asymptotes at $x = 0, -1, 1$.

(c) Since the derivative of $e^x$ is $e^x$, the derivative of $e^{-x}$ is $-e^{-x}$. Therefore the derivative of $-e^{-x}$ is $e^{-x}$. The general anti-derivative of $e^{-x}$ is therefore $h(x) = -e^{-x} + c$. Then $h(0) = -e^0 + c = -1 + c$, and setting this equal to 0 gives $c = 1$. Therefore $h(x) = -e^{-x} + 1$.

Question 2. (25 points)

(a) $f(0) = 1$, so the $y$-intercept is at $y = 1$. Solving $f(x) = 0$ gives $x = -1$, so the only $x$-intercept is at $x = -1$.

(b) The derivative is never 0, so there are no relative maxima or minima.

(c) The second derivative is positive if $(1-x)^3 > 0$, i.e. $1 - x > 0$, i.e. $x < 1$. Conversely it is negative if $x > 1$. Therefore it is concave up for $x < 1$ and concave down for $x > 1$.

(d) The second derivative is never 0: there are no inflection points.

(e) $\lim_{x\to\pm\infty} \frac{1+x}{1-x} = \lim_{x\to\pm\infty} \frac{x(\frac{1}{x}+1)}{x(\frac{1}{x}-1)} = \lim_{x\to\pm\infty} \frac{1}{-1} = -1$ So $y = -1$ is an asymptote to the right and left. There is a vertical asymptote at $x = 1$.

(f)

Question 3. (15 points)

The area of the rectangle is $A = 2xy$. To relate $x$ and $y$, by similar triangles $\frac{10}{2} = \frac{y}{2-x}$, so $y = 5(2-x)$. Therefore $A = 2x * 5(2-x) = 10x(2-x) = -10x^2 + 20x$. Note that $x$ can vary from 0 to 2. Then $A' = -20x + 20$, and setting this equal to 0 gives $20x = 20$, i.e. $x = 1$. The second derivative is $-20$, so this is a relative maximum. The endpoint $x = 0$ gives $A = 0$, and $x = 2$ also gives $A = 0$. So the maximum value is at $x = 1$. Then $y = 5(2-1) = 5$, and the area of the rectangle is $A = 2xy = 2 * 1 * 5 = 10$.

Question 4. (20 points)

(a) The derivative is $f'(x) = \frac{7}{3}x^{\frac{4}{3}}$, and the second derivative is $f''(x) = \frac{28}{9}x^{\frac{1}{3}}$. These are both zero only at $x = 0$. Since $f'(0) = 0$ this is a critical point, and could be a relative maximum or minimum. Since $f''(0) = 0$, this does not determine whether it is a relative maximum, minimum, or inflection point. However $f'(x) > 0$ for all $x \neq 0$, since $x^{\frac{4}{3}} = (x^{\frac{1}{3}})^4$, and since 4 is even this is always positive (unless $x = 0$). So the function is increasing for all $x \neq 0$, so it does not have a relative maximum or minimum of $x = 0$.

(b) If the graph of $g'(x)$ is increasing, then $g''(x) > 0$. Let $x = c$ be the point where $g'(x)$ is maximum. Then $g'(x)$ is increasing for $x < c$, and decreasing for $x > c$. Therefore the graph of $g(x)$ is concave down for $x > c$, and concave up for $0 < x < c$.