Math 140, Jeffrey Adams/Test III SOLUTIONS

Question 1. (25 points) (a)

The function x^3 has derivative $3x^2$, so the derivative of $\frac{1}{3}x^3$ is x^2. Also the derivative of $\cos(x)$ is $-\sin(x)$. So the derivative of $\frac{1}{3}x^3+\cos(x)$ is $x^2-\sin(x)$. Adding any constant c gives $f(x)=\frac{1}{3}x^3+\cos(x)+c$. Plugging in $x=0$ gives $0+1+c=3$, i.e. $c=2$, i.e. $f(x)=\frac{1}{3}x^3+\cos(x)+2$.

(b) $f'(x)=\sqrt{6-x}+x\frac{1}{2}(6-x)^{-\frac{1}{2}}(-1)=\sqrt{6-x}-\frac{x}{2\sqrt{6-x}}$. Setting this equal to 0 gives $\sqrt{6-x}-\frac{x}{2\sqrt{6-x}}=0$, or $\sqrt{6-x}=\frac{x}{2\sqrt{6-x}}$, i.e. $2(6-x)=x$, i.e. $12-2x=x$, or $3x=12$, so $x=4$. The value of the function at $x=4$ is $4\sqrt{2}$.

Question 2 (15 points) (a) Roughly, the 5 and 7 terms don't matter as $x\to\infty$, and then we can cancel the $\sqrt{x}$ terms, to give $\frac{12}{-3}=-4$. More precisely, multiply by $\frac{1}{\sqrt{x}}/\frac{1}{\sqrt{x}}$ to give $\frac{\frac{5}{\sqrt{x}}+12}{\frac{7}{\sqrt{x}}-3}$. The first terms in the numerator and denominator go to zero as $x\to\infty$, so the limit is $\frac{12}{-3}=-4$.

(b) The only terms which matter are the leading terms, i.e. the highest power of x. Multiplying out the denominator, the leading term is $x(2x)(3x)=6x^3$. The leading term in the numerator is cx^3. Therefore the limit is $\lim_{x\to\infty}\frac{cx^3+\dots}{6x^3+\dots}=\frac{c}{6}$. For this to equal 1, take $c=6$.

Question 3 (25 points)

(a) The vertical asymptotes are found where $\lim_{x\to c}=\infty$, which can only happen if the denominator is 0, i.e. $x=-2$. Since the numerator does not go to zero at $x=-2$, this is a vertical asymptote.

The limit, as $x\to\pm\infty$ is 0, since the power of x in the denominator is greater than the power in the numerator. So $x=0$ is a horizontal asymptote to the left and right.

The x intercept is given by setting $f(x)=0$, i.e. $x=0$. The y intercept is $f(0)=0$. So $(0,0)$ is the only x intercept, and is the y intercept.

Since $16x^2$ is always non-negative, the function is positive if $8+x^3$ is positive, and negative if $8+x^3$ is negative (except at 0, in which case $f(x)=0$. That is, $8+x^3>0$ if $x^3<-8$, i.e. $x<-2$. So $f(x)$ is positive to the right of $x=-2$ except it is 0 at $x=0$, and negative to the left of -2.

(3) Based on this information, the graph of $f(x)$ to the left of -2 is clear. It is positive to the right of -2, except at 0 where it is 0. Therefore 0 must be a min (it cannot cross into the negative region). Since it has the given asymptotes, it must look something like this:

(c) $f'(x) = 0$ at $x = 0$ and $x^3 = 16$, i.e. $x = \sqrt[3]{16}$. At 0 the second derivative is $\frac{32(64)}{8} > 0$, so 0 is a local minimum. At $\sqrt[3]{16}$ the second derivative is $\frac{32(64-56*16+16^2)}{(8+16)^3}$, which is negative. So $\sqrt[3]{16}$ is a local max.

(d) By the information on the second derivative, the graph is concave up at 1, down at 2 and 3, and up at 4. Therefore the inflection points are between 1 and 2, and between 3 and 4. Also there is a local max at $\sqrt[3]{16}$. Since $x^3 = 8$ and $3^3 = 27$, this is between 2 and 3. Therefore the graph looks like this:

Question 4 (15 points)

(a) Let $f(t)$ be the amount at time t. Then $f(t) = ce^{kt}$ for some c and k. The information given is that $f(15) = \frac{1}{4}f(0)$. Plugging in to $f(t)$ this gives $ce^{15k} = \frac{1}{4}ce^0$, or $e^{15k} = \frac{1}{4}$. Taking logs gives $\ln(e^{15k}) = \ln(\frac{1}{4})$, or $15k = \ln(\frac{1}{4}) = -\ln(4)$. Therefore $k = -\frac{\ln(4)}{15}$.

We are looking for the time t at which $f(t) = .05f(0)$. Plugging into the function, this gives $ce^{kt} = .05ce^0$, or $e^{kt} = .05$. Taking logs again gives $kt = \ln(.05)$, or $t = \frac{\ln(.05)}{k}$. Plugging in the value of k gives $t = \frac{\ln(.05)}{\frac{-\ln(4)}{15}} = -15\frac{\ln(.05)}{\ln(4)}$.

(b) We want to find c so that $f(20) = 12$. That is, $ce^{20k} = 12$, or $c = 12e^{-20k}$. Plugging in the value of k from (a) gives $c = 12e^{-20(-\ln(4)/15)}$. (This can be simplified to $12e^{\frac{4}{3}\ln(4)}$, or further to $12e^{\ln(4^{\frac{4}{3}})} = 124^{\frac{4}{3}} = 12\sqrt[3]{256} \simeq 76.2$)