Math 140, Jeffrey Adams
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SOLUTIONS

Question 1.

(a) $\frac{f(0)-f(-3)}{0-(-3)} = \frac{(-2)-(-27-3-2)}{3} = \frac{30}{3} = 10$, so $f'(x) = 10$. $f'(x) = 3x^2+1$, so $3x^2+1 = 10$, or $3x^2 = 9$, i.e. $x^2 = 3$, i.e. $x = \pm\sqrt{3}$. The value which is between -3 and 0 is $\boxed{x = -\sqrt{3}}$

(b) $\boxed{f(x) = \dfrac{1}{2}x^2 + \dfrac{1}{3}e^{3x} + C}$ Then $f(0) = 0 + \frac{1}{3} + C$, and setting this equal to 1 gives $\boxed{C = \dfrac{2}{3}}$

Question 2.

(a) Rough idea: ignore the constant terms to get $\frac{2x}{\sqrt{2x^2}} = \frac{2x}{\sqrt{2}x} = \sqrt{2}$. This is correct: the limit is $\boxed{\sqrt{2}}$

More carefully: multiply by $\frac{1/x}{1/\sqrt{x^2}}$ to get $\frac{2x+1}{\sqrt{2x^2+1}} = \frac{2+\frac{1}{x}}{\sqrt{2+\frac{1}{x^2}}}$. As $x \to \infty$ this clearly goes to $2/\sqrt{2} = \sqrt{2}$.

(b) Let $t = 0$ be 3 PM. The first ship's position is given by $x(t)$. The velocity is 3, so $x(t) = 3t + C$ for some C. Since $x(0) = 0$, $C = 0$, and $x(t) = 3t$.

The second ship's position is $y(t) = -4t + D$. Since $y(0) = 5$ this gives $y(0) = -4(0) + D = 5$, and $D = 5$. So $y(t) = -4t + 5$.

The distance $D(t)$ is $D(t) = \sqrt{(3t)^2 + (-4t+5)^2}$. This equals $\sqrt{9t^2 + 16t^2 - 40t + 25} = \sqrt{25t^2 - 40t + 25} = \sqrt{5}\sqrt{5t^2 - 8t + 5}$.

Take the derivative: $f'(t) = \sqrt{5}\frac{1}{2}(5t^2 - 8t + 5)^{-\frac{1}{2}}(10t - 8)$. Setting this equal to 0, we can ignore the $\sqrt{\ }$ terms, and set $10t - 8 = 0$, or $t = 4/5$. This says the time is 3 PM plus 4/5 of an hour, i.e. $\boxed{3:48 \quad \text{PM}}$

Question 3.

(a) The population is $P(t) = Ce^{kt}$. Since the doubling time is 3 years, this gives $P(3) = 2P(0)$, i.e. $Ce^{3k} = 2C$, or $e^{3k} = 2$, so $3k = \ln(2)$, or $k = \ln(2)/3$.
Setting $P(t) = 500,000 = 5C$ gives $P(t) = Ce^{kt} = 5C$, or $e^{kt} = 5$. Then $kt = \ln(5)$, so $t = \ln(5)/k$, i.e. $t = \ln(5)/(\ln(2)/3) = 3\ln(5)/\ln(2)$.

(b) $f'(x) = 3x^2 - 3$, and setting this equal to 0 gives $3x^2 = 3$, or $x = \pm 1$. Also check the endpoints:
$f(-2) = -8 + 6 = -2$, $f(3) = 27 - 9 = 18$, $f(1) = -2$ and $f(-1) = -1 + 3 = 2$. The max is $f(3) = 18$, and the mins are $f(-2) = f(1) = -2$.

Question 4. Consider the function $f(x) = \dfrac{1+x}{1-x}$.

(a) The limit at $x \to \pm\infty$ is -1, so it has horizontal asymptote -1 to the left and right. It has a vertical asymptote at $x = 1$.

(b) $f'(x) = \frac{1(1-x)-(1+x)(-1)}{(1+x)^2} = \frac{2}{(1-x)^2}$, which is always positive: the function is increasing everywhere.

(c) The second derivative is $-4(1-x)^{-3}(-1) = \frac{4}{(1-x)^3}$. This is positive if $1 - x < 0$, i.e. $x < 1$, and negative if $x > 1$. So $f(x)$ is concave up for $x < 1$, and concave down for $x > 1$.

(d) Sketch the function.