**Math 141, Jeffrey Adams**

Test 4, December 8, 2000 SOLUTIONS

Question 1. (a) Compute $f'(x) = \ln(3)3^x$. The surface area is

$$\int_0^3 2\pi 3^x \sqrt{1 + (\ln(3)3^x)^2}\, dx$$

(b) Method 1. Solve for y: $y^2 = 1 - 4x^2$, or $y = \pm\sqrt{1-4x^2}$. The top half of the ellipse is given by $f(x) = \sqrt{1-4x^2}$. Then $f'(x) = \frac{1}{2}(1-4x^2)^{-\frac{1}{2}}(-8x) = \frac{-4x}{\sqrt{1-4x^2}}$. The x-intercepts are given by (take $y = 0$) $4x^2 = 1$ or $x = \pm\frac{1}{2}$. The length of the ellipse is twice length of the top half, which is the graph of $f(x)$. So the length is $2\int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{1 + \frac{16x^2}{1-4x^2}}\, dx$.

Method 2. Parametrize the curve by $x(t) = \frac{1}{2}\cos(t), y(t) = \sin(t)$ with $0 \le t \le 2\pi$. Then $x'(t) = -\frac{1}{2}\sin(t), y'(t) = \cos(t)$, and the integral is $\int_0^{2\pi} \sqrt{\frac{1}{4}\sin^2(t) + \cos^2(t)}\, dt$.

Question 2

(a)(a) The integral is $\int_0^1 \pi((x^2)-(x^2)^2)\, dx = \pi\int_0^1 (x^2 - x^4)\, dx = \frac{\pi}{3}x^3 - \frac{\pi}{5}x^5|_0^1 = \frac{\pi}{3} - \frac{\pi}{5} = \frac{2\pi}{15}$.

(b) Put the origin at the bottom of the tank, with the positive x-axis in pointing up. The slice of the tank at height $x$ is a disk. By similar triangles the radius of the disk $r(x)$ is given by $\frac{2}{4} = \frac{r(x)}{x}$, or $r(x) = \frac{1}{2}x$. The area of this slice is therefore $\pi(\frac{1}{2}x)^2 = \frac{\pi}{4}x^2$. The weight of the slice at $x$ of thickness $dx$ is therefore $62.5\frac{\pi}{4}x^2\, dx$. This slice has to be raised $(4-x)+2 = 6-x$ feet. The water goes from $x = 0$ to $x = 3$. The integral is therefore $\int_0^3 62.5(6-x)(\frac{\pi}{4}x^2)\, dx = 62.5\frac{\pi}{4}\int_0^3 (6-x)x^2\, dx$.

Alternatively if you put the origin at the top of the tank with the positive x-axis pointing down, the radius at $x$ is $\frac{1}{2}(4-x)$. The integral is $62.5\frac{\pi}{4}\int_4^1 (2+x)(4-x)^2\, dx$. Note that the change of variable $w = 4-x$ converts this integral into the previous one.

Question 3

(a) $x'(t) = 2\sin(3t)\cos(3t)3 = 6\sin(3t)\cos(3t)$, and $y'(t) = -2\sin(2t)$. The length of the curve is therefore $\int_1^2 \sqrt{36\sin^2(3t)\cos^2(3t) + 4\sin^2(2t)}\, dt$.

(b)

$$\begin{aligned}
\int_{-\pi/4}^{\pi/4} \frac{1}{2}[3\cos(2\theta)]^2\, d\theta &= \int_{-\pi/4}^{\pi/4} \frac{9}{2}\cos^2(2\theta)\, d\theta \\
&= \frac{9}{2}\int_{-\pi/4}^{\pi/4} \frac{1}{2}(1+\cos(4\theta))\, d\theta \\
&= \frac{9}{4}[\theta + \frac{1}{4}\sin(4\theta)]_{-\pi/4}^{\pi/4} \\
&= \frac{9}{4}[\frac{\pi}{4}+\frac{\pi}{4}] + \frac{9}{16}[\sin(\pi)-\sin(-\pi)] \\
&= \frac{9}{4}\frac{\pi}{2} - 0 = \frac{9\pi}{8}
\end{aligned}$$

Question 4

(a) We have $\overline{z} = 4+4i$, $z\overline{z} = (4+4i)(4-4i) = 16+16 = 32$, $|z| = \sqrt{16+16} = \sqrt{32} = 4\sqrt{2}$. If $4-4i = re^{i\theta}$ then $r = |z| = 4\sqrt{2}$ and $\tan(\theta) = \frac{-4}{4} = -1$. Since the point is in the fourth quadrant this gives $\theta = -\frac{\pi}{4}$. So $4-4i = 4\sqrt{2}e^{-\pi i/4}$.

(b) [15] Write $-8 = 8e^{i\pi}$. Write $z = re^{i\theta}$, so $r^3e^{3i\theta} = -8 = 8e^{i\pi}$. We can add multiples of $2\pi$ on the right hand side, to get $r^3e^{3i\theta} = 8e^{i(1+2k)\pi}$. Therefore $r^3 = 8$, i.e. $r = 2$ and $3\theta = (1+2k)\pi$, or $\theta = \frac{1+2k}{3}\pi$. If $k = 0,1,2$ we get $\theta = \frac{\pi}{3}, \frac{3\pi}{3} = \pi$, and $\frac{5\pi}{3}$ respectively. So $z = 2e^{i\frac{\pi}{3}} = 2(\frac{1}{2} + i\frac{\sqrt{3}}{2}) = 1+\sqrt{3}i$, $2e^{i\pi} = -2$ and $2e^{\frac{5\pi}{3}} = 2(\cos(\frac{5\pi}{3}) + i\sin(\frac{5\pi}{3})) = 2(\frac{1}{2} - i\frac{\sqrt{3}}{2}) = 1-\sqrt{3}i$.

## FINAL INFORMATION

There is a review for the final on Thursday, December 11, in Room 0226 HJ Patterson Hall, 4-6 PM

The final is on Monday, December 15, 1:30-3:30. The room assignments are:

Xia Wang (0111/0121): PLS (Plant Sciences Building) 1140
Hector Gomez (0112/0122): CCC (Cambridge Community Center) 1205
Brian Alford (0131): HHP (Health and Human Performances Bulding) 1312
Eric Errthurm (0141): KEY (Frances Scott Key Hall) 0103