Third In-Class Exam Solutions
Math 246, Professor David Levermore
Tuesday, 24 April 2018

(1) [6] Recast the ordinary differential equation $y'''' = e^y y''' + (y'')^2 + \cos(t^3 + y')$ as a first-order system of ordinary differential equations.

Solution. Because the equation is fourth order, the first-order system must have dimension at least four. The simplest such first-order system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ e^{x_1} x_4 + (x_3)^2 + \cos(t^3 + x_2) \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} y \\ y' \\ y'' \\ y''' \end{pmatrix}.$$

(2) [10] Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} 4 \\ 3t^2 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 1 + t^4 \end{pmatrix}$.

(a) [2] Compute the Wronskian $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t)$.

(b) [3] Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ wherever $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

(c) [2] Give a general solution to the system found in part (b).

(d) [3] Compute the Green matrix associated with the system found in part (b).

Solution (a). The Wronskian is

$$\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix} 4 & t^2 \\ 3t^2 & 1 + t^4 \end{pmatrix} = 4 \cdot (1 + t^4) - 3t^2 \cdot t^2 = 4 + t^4 .$$

Solution (b). Let $\mathbf{\Psi}(t) = \begin{pmatrix} 4 & t^2 \\ 3t^2 & 1 + t^4 \end{pmatrix}$. Because $\mathbf{\Psi}'(t) = \mathbf{A}(t)\mathbf{\Psi}(t)$, we have

$$\begin{aligned} \mathbf{A}(t) = \mathbf{\Psi}'(t)\mathbf{\Psi}(t)^{-1} &= \begin{pmatrix} 0 & 2t \\ 6t & 4t^3 \end{pmatrix} \begin{pmatrix} 4 & t^2 \\ 3t^2 & 1 + t^4 \end{pmatrix}^{-1} \\ &= \frac{1}{4 + t^4} \begin{pmatrix} 0 & 2t \\ 6t & 4t^3 \end{pmatrix} \begin{pmatrix} 1 + t^4 & -t^2 \\ -3t^2 & 4 \end{pmatrix} = \frac{1}{4 + t^4} \begin{pmatrix} -6t^3 & 8t \\ 6t - 6t^5 & 10t^3 \end{pmatrix} . \end{aligned}$$

Solution (c). A general solution is

$$\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) = c_1 \begin{pmatrix} 4 \\ 3t^2 \end{pmatrix} + c_2 \begin{pmatrix} t^2 \\ 1 + t^4 \end{pmatrix} .$$

Solution (d). By using the fundamental matrix $\mathbf{\Psi}(t)$ from part (b) we find that the Green matrix is

$$\begin{aligned} \mathbf{G}(t, s) = \mathbf{\Psi}(t)\mathbf{\Psi}(s)^{-1} &= \begin{pmatrix} 4 & t^2 \\ 3t^2 & 1 + t^4 \end{pmatrix} \begin{pmatrix} 4 & s^2 \\ 3s^2 & 1 + s^4 \end{pmatrix}^{-1} \\ &= \frac{1}{4 + s^4} \begin{pmatrix} 4 & t^2 \\ 3t^2 & 1 + t^4 \end{pmatrix} \begin{pmatrix} 1 + s^4 & -s^2 \\ -3s^2 & 4 \end{pmatrix} \\ &= \frac{1}{4 + s^4} \begin{pmatrix} 4 + 4s^4 - 3t^2 s^4 & 4t^2 - 4s^2 \\ 3t^2 + 3t^2 s^4 - 3t^2 - 3t^4 s^2 & 4 + 4t^4 - 3t^2 s^2 \end{pmatrix} . \end{aligned}$$

(3) [6] Two interconnected tanks are filled with brine (salt water). At $t = 0$ the first tank contains 26 liters and the second contains 19 liters. Brine with a salt concentration of 5 grams per liter flows into the first tank at 6 liters per hour. Well-stirred brine flows from the first tank into the second at 8 liters per hour, from the second into the first at 7 liters per hour, from the first into a drain at 3 liter per hour, and from the second into a drain at 2 liters per hour. At $t = 0$ there are 17 grams of salt in the first tank and 31 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

Solution. Let $V_1(t)$ and $V_2(t)$ be the volumes (lit) of brine in the first and second tank at time t hours. Let $S_1(t)$ and $S_2(t)$ be the mass (gr) of salt in the first and second tank at time t hours. Because the mixtures are assumed to be well-stirred, the salt concentration of the brine in the tanks at time t are $C_1(t) = S_1(t)/V_1(t)$ and $C_2(t) = S_2(t)/V_2(t)$ respectively. In particular, these are the concentrations of the brine that flows out of these tanks. We have the following picture.

5 gr/lit 6 lit/hr	$\rightarrow$	$V_1(t)$ lit $S_1(t)$ gr	$\rightarrow$	$C_1(t)$ gr/lit 8 lit/hr	$\rightarrow$	$V_2(t)$ lit $S_2(t)$ gr		
$C_1(t)$ gr/lit 3 lit/hr	$\leftarrow$		$\leftarrow$	$C_2(t)$ gr/lit 7 lit/hr	$\leftarrow$		$\rightarrow$	$C_2(t)$ gr/lit 2 lit/hr
		$V_1(0) = 26$ lit $S_1(0) = 17$ gr				$V_2(0) = 19$ lit $S_2(0) = 31$ gr		

We are asked to write down an initial-value problem that governs $S_1(t)$ and $S_2(t)$.

The rates work out so there will be $V_1(t) = 26 + 2t$ liters of brine in the first tank and $V_2(t) = 19 - t$ liters in the second. Then $S_1(t)$ and $S_2(t)$ are governed by the initial-value problem

$$\begin{aligned} \frac{\mathrm{d}S_1}{\mathrm{d}t} &= 5 \cdot 6 + \frac{S_2}{19 - t}\,7 - \frac{S_1}{26 + 2t}\,8 - \frac{S_1}{26 + 2t}\,3\,, && S_1(0) = 17\,, \\ \frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{S_1}{26 + 2t}\,8 - \frac{S_2}{19 - t}\,7 - \frac{S_2}{19 - t}\,2\,, && S_2(0) = 31\,. \end{aligned}$$

Your answer could be left in the above form. However, it can be simplified to

$$\begin{aligned} \frac{\mathrm{d}S_1}{\mathrm{d}t} &= 30 + \frac{7}{19 - t}\,S_2 - \frac{11}{26 + 2t}\,S_1\,, && S_1(0) = 17\,, \\ \frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{8}{26 + 2t}\,S_1 - \frac{9}{19 - t}\,S_2\,, && S_2(0) = 31\,. \end{aligned}$$

Remark. This first-order system of differential equations is *linear*. Its coefficients are undefined either at $t = -13$ or at $t = 19$ and are continuous elsewhere. Its forcing is constant, so is continuous everywhere. Therefore the natural interval of definition for the solution of this initial-value problem is $(-13, 19)$ because:

- the initial time $t = 0$ is in $(-13, 19)$;
- all the coefficients and the forcing are continuous over $(-13, 19)$;
- two coefficients are undefined at $t = -13$;
- two coefficients are undefined at $t = 19$.

However, it could also be argued that the interval of definition for the solution of this initial-value problem is $[0, 19)$ because the word problem starts at $t = 0$.

(4) [10] Solve the initial-value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Solution by Formula. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} -1 & 1 \\ -4 & -5 \end{pmatrix}$ is

$$p(z) = z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 6z + 9 = (z+3)^2.$$

This is a perfect square with $\mu = -3$. Then

$$\begin{aligned} e^{t\mathbf{A}} = e^{-3t}\left[\mathbf{I} + t\left(\mathbf{A} + 3\mathbf{I}\right)\right] &= e^{-3t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\right] \\ &= e^{-3t}\begin{pmatrix} 1+2t & t \\ -4t & 1-2t \end{pmatrix}. \end{aligned}$$

(Check that $\mathrm{tr}(\mathbf{A} - 4\mathbf{I}) = 0$!) Therefore the solution of the initial-value problem is

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}^I = e^{-3t}\begin{pmatrix} 1+2t & t \\ -4t & 1-2t \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = e^{4t}\begin{pmatrix} 2+4t \\ -8t \end{pmatrix}.$$

(5) [6] Given that 2 is an eigenvalue of the matrix

$$\mathbf{B} = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 2 & -3 \\ 0 & 2 & 10 \end{pmatrix},$$

find all the eigenvectors of $\mathbf{B}$ associated with 2.

Solution. The eigenvectors of $\mathbf{B}$ associated with 2 are all nonzero vectors $\mathbf{v}$ such that $\mathbf{B}\mathbf{v} = 2\mathbf{v}$. Equivalently, they are all nonzero vectors $\mathbf{v}$ such that $(\mathbf{B} - 2\mathbf{I})\mathbf{v} = \mathbf{0}$, which is

$$\begin{pmatrix} -2 & -1 & 2 \\ 1 & 0 & -3 \\ 0 & 2 & 8 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The entries of $\mathbf{v}$ thereby satisfy the homogeneous linear algebraic system

$$\begin{aligned} -2v_1 - v_2 + 2v_3 &= 0, \\ v_1 \qquad - 3v_3 &= 0, \\ 2v_2 + 8v_3 &= 0. \end{aligned}$$

This system may be solved either by elimination or by row reduction. By either method its general solution is found to be

$$v_1 = 3\alpha, \quad v_2 = -4\alpha, \quad v_3 = \alpha, \qquad \text{for any constant } \alpha.$$

Therefore the eigenvectors of $\mathbf{B}$ associated with 2 each have the form

$$\alpha\begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} \qquad \text{for some constant } \alpha \neq 0.$$

(6) [8] A 4×4 matrix $\mathbf{C}$ has the eigenpairs

$$\left(5\,,\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right), \qquad \left(2\,,\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}\right), \qquad \left(-1\,,\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}\right), \qquad \left(-4\,,\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}\right).$$

(a) Give an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ such that $e^{t\mathbf{C}} = \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1}$. (You do not have to compute either $\mathbf{V}^{-1}$ or $e^{t\mathbf{C}}$!)

(b) Give a fundamental matrix for the system $\mathbf{x}' = \mathbf{C}\mathbf{x}$.

Solution (a). One choice for $\mathbf{V}$ and $\mathbf{D}$ is

$$\mathbf{V} = \begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix}5 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -4\end{pmatrix}.$$

Solution (b). Use the given eigenpairs to construct the real eigensolutions

$$\mathbf{x}_1(t) = e^{5t}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \qquad \mathbf{x}_2(t) = e^{2t}\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix},$$

$$\mathbf{x}_3(t) = e^{-t}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}, \qquad \mathbf{x}_4(t) = e^{-4t}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}.$$

Then a fundamental matrix for the system is

$$\mathbf{\Psi}(t) = \begin{pmatrix}\mathbf{x}_1(t) & \mathbf{x}_2(t) & \mathbf{x}_3(t) & \mathbf{x}_4(t)\end{pmatrix} = \begin{pmatrix}e^{5t} & e^{2t} & e^{-t} & e^{-4t}\\ e^{5t} & e^{2t} & -e^{-t} & -e^{-4t}\\ e^{5t} & -e^{2t} & e^{-t} & -e^{-4t}\\ e^{5t} & -e^{2t} & -e^{-t} & e^{-4t}\end{pmatrix}.$$

Alternative Solution (b). Given the $\mathbf{V}$ and $\mathbf{D}$ from part (a), a fundamental matrix for the system is

$$\mathbf{\Psi}(t) = \mathbf{V}e^{t\mathbf{D}} = \begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix}\begin{pmatrix}e^{5t} & 0 & 0 & 0\\ 0 & e^{2t} & 0 & 0\\ 0 & 0 & e^{-t} & 0\\ 0 & 0 & 0 & e^{-4t}\end{pmatrix}$$

$$= \begin{pmatrix}e^{5t} & e^{2t} & e^{-t} & e^{-4t}\\ e^{5t} & e^{2t} & -e^{-t} & -e^{-4t}\\ e^{5t} & -e^{2t} & e^{-t} & -e^{-4t}\\ e^{5t} & -e^{2t} & -e^{-t} & e^{-4t}\end{pmatrix}.$$

(7) [8] Find a real general solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -4 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Solution by Formula. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} -3 & 2 \\ -4 & -7 \end{pmatrix}$ is

$$p(z) = z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 10z + 29 = (z+5)^2 + 2^2 .$$

This is a sum of squares with $\mu = -5$ and $\nu = 2$. Then

$$\begin{aligned} e^{t\mathbf{A}} &= e^{-5t}\left[\cos(2t)\mathbf{I} + \frac{\sin(2t)}{2}(\mathbf{A}+5\mathbf{I})\right] \\ &= e^{-5t}\left[\cos(2t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(2t)}{2}\begin{pmatrix} 2 & 2 \\ -4 & -2 \end{pmatrix}\right] \\ &= e^{-5t}\begin{pmatrix} \cos(2t)+\sin(2t) & \sin(2t) \\ -2\sin(2t) & \cos(2t)-\sin(2t) \end{pmatrix}. \end{aligned}$$

(Check that $\mathrm{tr}(\mathbf{A}+5\mathbf{I}) = 0$!) Therefore a real general solution of the system is

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{c} = c_1 e^{-5t}\begin{pmatrix} \cos(2t)+\sin(2t) \\ -2\sin(2t) \end{pmatrix} + c_2 e^{-5t}\begin{pmatrix} \sin(2t) \\ \cos(2t)-\sin(2t) \end{pmatrix}.$$

Solution by Eigen Methods. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} -3 & 2 \\ -4 & -7 \end{pmatrix}$ is

$$p(z) = z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 10z + 29 = (z+5)^2 + 2^2 .$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $-5+i2$ and $-5-i2$. Consider the matrix

$$\mathbf{A} - (-5-i2)\mathbf{I} = \begin{pmatrix} 2+i2 & 2 \\ -4 & -2+i2 \end{pmatrix}.$$

After checking that the determinant of this matrix is zero, we can read off from its second column that an eigenpair of $\mathbf{A}$ is

$$\left(-5+i2\,,\ \begin{pmatrix} 1 \\ -1+i \end{pmatrix}\right).$$

(Another eigenpair is the complex conjugate of this one, but we will not need it.) This eigenpair yields the complex-valued eigensolution

$$\begin{aligned} \mathbf{x}(t) &= e^{(-5+i2)t}\begin{pmatrix} 1 \\ -1+i \end{pmatrix} = e^{-5t}\big(\cos(2t) + i\sin(2t)\big)\begin{pmatrix} 1 \\ -1+i \end{pmatrix} \\ &= e^{-5t}\begin{pmatrix} \cos(2t)+i\sin(2t) \\ \big(\cos(2t)+i\sin(2t)\big)(-1+i) \end{pmatrix} \\ &= e^{-5t}\begin{pmatrix} \cos(2t)+i\sin(2t) \\ \big(-\cos(2t)-\sin(2t)\big) + i\big(\cos(2t)-\sin(2t)\big) \end{pmatrix}. \end{aligned}$$

From the real and imaginary parts of this complex-valued eigensolution we can read off that a fundamental set of real solutions is

$$\mathbf{x}_1(t) = e^{-5t}\begin{pmatrix} \cos(2t) \\ -\cos(2t) - \sin(2t) \end{pmatrix}, \qquad \mathbf{x}_2(t) = e^{-5t}\begin{pmatrix} \sin(2t) \\ \cos(2t) - \sin(2t) \end{pmatrix}.$$

Therefore a real general solution is

$$\mathbf{x}(t) = c_1 e^{-5t}\begin{pmatrix} \cos(2t) \\ -\cos(2t) - \sin(2t) \end{pmatrix} + c_2 e^{-5t}\begin{pmatrix} \sin(2t) \\ \cos(2t) - \sin(2t) \end{pmatrix}.$$

(8) [8] Find a real general solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Solution by Eigen Methods. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 3 & 5 \\ 3 & 1 \end{pmatrix}$ is

$$p(z) = z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 4z - 12 = (z-6)(z+2).$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are -2 and 6. Consider the matrices

$$\mathbf{A} + 2\mathbf{I} = \begin{pmatrix} 5 & 5 \\ 3 & 3 \end{pmatrix}, \qquad \mathbf{A} - 6\mathbf{I} = \begin{pmatrix} -3 & 5 \\ 3 & -5 \end{pmatrix}.$$

After checking that the determinant of each matrix is zero, we can read off that eigenpairs of $\mathbf{A}$ are

$$\left(-2, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right), \qquad \left(6, \begin{pmatrix} 5 \\ 3 \end{pmatrix}\right).$$

Therefore a real general solution of the system is

$$\mathbf{x}(t) = c_1 e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{6t}\begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

Solution by Formula. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 3 & 5 \\ 3 & 1 \end{pmatrix}$ is

$$p(z) = z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 4z - 12 = (z-2)^2 - 4^2.$$

This is a difference of squares with $\mu = 2$ and $\nu = 4$. Then

$$\begin{aligned} e^{t\mathbf{A}} &= e^{2t}\left[\cosh(4t)\mathbf{I} + \frac{\sinh(4t)}{4}\big(\mathbf{A} - 2\mathbf{I}\big)\right] \\ &= e^{2t}\left[\cosh(4t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sinh(4t)}{4}\begin{pmatrix} 1 & 5 \\ 3 & -1 \end{pmatrix}\right] \\ &= e^{2t}\begin{pmatrix} \cosh(4t) + \frac{1}{4}\sinh(4t) & \frac{5}{4}\sinh(4t) \\ \frac{3}{4}\sinh(4t) & \cosh(4t) - \frac{1}{4}\sinh(4t) \end{pmatrix}. \end{aligned}$$

(Check that $\operatorname{tr}(\mathbf{A}-2\mathbf{I})=0$!) Therefore a real general solution of the system is

$$\begin{aligned}\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{c} = e^{2t}\begin{pmatrix}\cosh(4t)+\frac{1}{4}\sinh(4t) & \frac{5}{4}\sinh(4t)\\ \frac{3}{4}\sinh(4t) & \cosh(4t)-\frac{1}{4}\sinh(4t)\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}\\ = c_1 e^{2t}\begin{pmatrix}\cosh(4t)+\frac{1}{4}\sinh(4t)\\ \frac{3}{4}\sinh(4t)\end{pmatrix} + c_2 e^{2t}\begin{pmatrix}\frac{5}{4}\sinh(4t)\\ \cosh(4t)-\frac{1}{4}\sinh(4t)\end{pmatrix}.\end{aligned}$$

(9) [10] Find the natural fundamental set of solutions associated with the initial-time 0 for the operator $\mathrm{L} = \mathrm{D}^3 + 4\mathrm{D}$.

Solution from Green Function. The operator $\mathrm{L} = \mathrm{D}^3 + 4\mathrm{D}$ has characteristic polynomial

$$p(s) = s^3 + 4s = s(s^2+4)\,.$$

We have the partial-fraction identity

$$\frac{1}{p(s)} = \frac{1}{s^3+4s} = \frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s}{s^2+4}\,.$$

Then from item 1 in the table with $a=0$ and $n=0$, and item 2 in the table with $a=0$ and $b=2$ we see that the Green function for the operator $\mathrm{L} = \mathrm{D}^3 + 4\mathrm{D}$ is

$$\begin{aligned}g(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right](t) &= \tfrac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s}\right](t) - \tfrac{1}{4}\mathcal{L}^{-1}\left[\frac{s}{s^2+2^2}\right](t)\\ &= \tfrac{1}{4}\cdot 1 - \tfrac{1}{4}\cos(2t)\,.\end{aligned}$$

Because we see the characteristic polynomial as

$$p(s) = s^3 + 0s^2 + 4s + 0\,,$$

the natural fundamental set is given by

$$\begin{aligned}N_2(t) &= g(t) = \tfrac{1}{4} - \tfrac{1}{4}\cos(2t)\,,\\ N_1(t) &= N_2'(t) + 0g(t) = \tfrac{1}{2}\sin(2t)\,,\\ N_0(t) &= N_1'(t) + 4g(t) = \cos(2t) + 4\big(\tfrac{1}{4} - \tfrac{1}{4}\cos(2t)\big) = 1\,.\end{aligned}$$

Solution from General Initial-Value Problem. For the operator $\mathrm{L} = \mathrm{D}^3 + 4\mathrm{D}$ the general initial-value problem for initial-time 0 is

$$y''' + 4y' = 0\,, \qquad y(0) = y_0\,, \quad y'(0) = y_1\,, \quad y''(0) = y_2\,.$$

Its characteristic polynomial is

$$p(z) = z^3 + 4z = z(z^2+4) = z(z^2+2^2)\,,$$

which has roots $i2$, $-i2$ and 0. Therefore a real general solution is

$$y(t) = c_1\cos(2t) + c_2\sin(2t) + c_3\,.$$

Because

$$y'(t) = -2c_1\sin(2t) + 2c_2\cos(2t)\,, \qquad y''(t) = -4c_1\cos(2t) - 4c_2\sin(2t)\,,$$

the general initial conditions yield the linear algebraic system

$$\begin{aligned} y_0 &= y(0) = c_1 \cos(0) + c_2 \sin(0) + c_3 = c_1 + c_3 \,, \\ y_1 &= y'(0) = -2c_1 \sin(0) + 2c_2 \cos(0) = 2c_2 \,, \\ y_2 &= y''(0) = -4c_1 \cos(0) - 4c_2 \sin(0) = -4c_1 \,. \end{aligned}$$

The solution of this system is

$$c_1 = -\tfrac{1}{4} y_2 \,, \qquad c_2 = \tfrac{1}{2} y_1 \,, \qquad c_3 = y_0 + \tfrac{1}{4} y_2 \,.$$

Therefore the solution of the general initial-value problem is

$$\begin{aligned} y &= -\tfrac{1}{4} y_2 \cos(2t) + \tfrac{1}{2} y_1 \sin(2t) + (y_0 + \tfrac{1}{4} y_2) \\ &= y_0 + y_1 \tfrac{1}{2} \sin(2t) + y_2 \tfrac{1}{4} \big(1 - \cos(2t)\big) \,. \end{aligned}$$

We can read off from this that the natural fundamental set of solutions associated with the initial-time 0 for the operator $\mathrm{L} = \mathrm{D}^3 + 4\mathrm{D}$ is

$$N_0(t) = 1 \,, \qquad N_1(t) = \tfrac{1}{2} \sin(2t) \,, \qquad N_2(t) = \tfrac{1}{4} \big(1 - \cos(2t)\big) \,.$$

(10) [8] Compute the Laplace transform of $f(t) = u(t-5)\, e^{-3t}$ from its definition. (Here u is the unit step function.)

Solution. The definition of Laplace transform gives

$$\mathcal{L}[f](s) = \lim_{T \to \infty} \int_0^T e^{-st} u(t-5)\, e^{-3t} \, \mathrm{d}t = \lim_{T \to \infty} \int_5^T e^{-(s+3)t} \, \mathrm{d}t \,.$$

When $s \leq -3$ this limit diverges to $+\infty$ because in that case we have for every $T > 5$

$$\int_5^T e^{-(s+3)t} \, \mathrm{d}t \geq \int_5^T \mathrm{d}t = T - 5 \,,$$

which clearly diverges to $+\infty$ as $T \to \infty$.

When $s > -3$ we have for every $T > 5$

$$\int_5^T e^{-(s+3)t} \, \mathrm{d}t = -\frac{e^{-(s+3)t}}{s+3} \bigg|_5^T = -\frac{e^{-(s+3)T}}{s+3} + \frac{e^{-(s+3)5}}{s+3} \,,$$

whereby

$$\mathcal{L}[f](s) = \lim_{T \to \infty} \left[-\frac{e^{-(s+3)T}}{s+3} + \frac{e^{-(s+3)5}}{s+3} \right] = \frac{e^{-(s+3)5}}{s+3} \qquad \text{for } s > -3 \,.$$

Therefore the definition of the Laplace transform shows that

$$\mathcal{L}[f](s) = \begin{cases} \dfrac{e^{-(s+3)5}}{s+3} & \text{for } s > -3 \,, \\ \text{undefined} & \text{for } s \leq -3 \,. \end{cases}$$

(11) [12] Consider the following (old style) MATLAB commands.

```
>> syms t s Y; f = ['t^2 + heaviside(t − 2)*(4 − t^2) − heaviside(t − 6)*4'];
>> diffeqn = sym('D(D(y))(t) + 6*D(y)(t) + 34*y(t) = ' f);
>> eqntrans = laplace(diffeqn, t, s);
>> algeqn = subs(eqntrans, {'laplace(y(t),t,s),t,s)', 'y(0)', 'D(y)(0)'}, {Y, 4, −2});
>> ytrans = simplify(solve(algeqn, Y));
>> y = ilaplace(ytrans, s, t)
```

(a) [4] Give the initial-value problem for $y(t)$ that is being solved.
(b) [8] Find the Laplace transform $Y(s)$ of the solution $y(t)$. (DO NOT take the inverse Laplace transform of $Y(s)$ to find $y(t)$, just solve for $Y(s)$!)

You may refer to the table on the last page.

Solution (a). The initial-value problem for $y(t)$ that is being solved is

$$y'' + 6y' + 34y = f(t)\,, \qquad y(0) = 4\,, \quad y'(0) = -2\,,$$

where the forcing $f(t)$ can be expressed either as the piecewise-defined function

$$f(t) = \begin{cases} t^2 & \text{for } 0 \le t < 2\,, \\ 4 & \text{for } 2 \le t < 6\,, \\ 0 & \text{for } 6 \le t\,, \end{cases}$$

or in terms of the unit step function as

$$f(t) = t^2 + u(t-2)(4-t^2) - u(t-6)4\,.$$

Solution (b). The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) + 6\mathcal{L}[y'](s) + 34\mathcal{L}[y](s) = \mathcal{L}[f](s)\,.$$

Because

$$\begin{aligned} \mathcal{L}[y](s) &= Y(s)\,, \\ \mathcal{L}[y'](s) &= s\,\mathcal{L}[y](s) - y(0) = s\,Y(s) - 4\,, \\ \mathcal{L}[y''](s) &= s\,\mathcal{L}[y'](s) - y'(0) = s^2Y(s) - 4s + 2\,, \end{aligned}$$

the Laplace transform of the initial-value problem becomes

$$\big(s^2Y(s) - 4s + 2\big) + 6\big(sY(s) - 4\big) + 34Y(s) = \mathcal{L}[f](s)\,.$$

This simplifies to

$$(s^2 + 6s + 34)Y(s) - 4s - 22 = \mathcal{L}[f](s)\,,$$

whereby

$$Y(s) = \frac{1}{s^2 + 6s + 34}\big(4s + 22 + \mathcal{L}[f](s)\big)\,.$$

To compute $\mathcal{L}[f](s)$, we write $f(t)$ as

$$\begin{aligned} f(t) &= t^2 + u(t-2)(4-t^2) - u(t-6)4 \\ &= t^2 + u(t-2)j_1(t-2) + u(t-6)j_2(t-6)\,, \end{aligned}$$

where by setting $j_1(t-2) = 4 - t^2$ and $j_2(t-6) = -4$ we see by the shifty step method that

$$j_1(t) = 4 - (t+2)^2 = 4 - t^2 - 4t - 4 = -t^2 - 4t\,, \qquad j_2(t) = -4\,.$$

Referring to the table on the last page, item 1 with $a = 0$ and $n = 0$, with $a = 0$ and $n = 1$, and with $a = 0$ and $n = 2$ shows that

$$\mathcal{L}[1](s) = \frac{1}{s}\,, \qquad \mathcal{L}[t](s) = \frac{1}{s^2}\,, \qquad \mathcal{L}[t^2](s) = \frac{2}{s^3}\,,$$

whereby item 6 with $c = 2$ and $j(t) = j_1(t)$ and with $c = 6$ and $j(t) = j_2(t)$ shows that

$$\mathcal{L}\big[u(t-2)j_1(t-2)\big](s) = e^{-2s}\mathcal{L}[j_1](s) = -e^{-2s}\mathcal{L}[t^2+4t](s) = -e^{-2s}\left(\frac{3}{s} - \frac{1}{s^2}\right)\,,$$

$$\mathcal{L}\big[u(t-6)j_2(t-6)\big](s) = e^{-6s}\mathcal{L}[j_2](s) = -e^{-6s}\mathcal{L}[6](s) = -e^{-6s}\frac{6}{s}\,.$$

Therefore

$$\begin{aligned}\mathcal{L}[f](s) &= \mathcal{L}\big[t^2 + u(t-2)j_1(t-2) + u(t-6)j_2(t-6)\big](s)\\ &= \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2}\right) - e^{-6s}\frac{4}{s}\,.\end{aligned}$$

Upon placing this result into the expression for $Y(s)$ found earlier, we obtain

$$Y(s) = \frac{1}{s^2+6s+34}\left(4s + 22 + \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2}\right) - e^{-6s}\frac{4}{s}\right)\,.$$

(12) [8] Find the inverse Laplace transform $\mathcal{L}^{-1}[X(s)](t)$ of the function

$$X(s) = e^{-4s}\,\frac{3s+11}{s^2+6s+13}\,.$$

You may refer to the table on the last page.

Solution. Referring to the table on the last page, item 6 with $c = 4$ implies that

$$\mathcal{L}^{-1}\big[e^{-4s}\,J(s)\big] = u(t-4)j(t-4)\,, \qquad \text{where} \qquad j(t) = \mathcal{L}^{-1}[J(s)](t)\,.$$

We apply this formula to

$$J(s) = \frac{3s+11}{s^2+6s+13}\,.$$

Because $s^2 + 6s + 13 = (s+3)^2 + 2^2$, we have the partial fraction identity

$$J(s) = \frac{3s+11}{s^2+6s+13} = \frac{3(s+3)+2}{(s+3)^2+2^2} = 3\,\frac{s+3}{(s+3)^2+2^2} + \frac{2}{(s+3)^2+2^2}\,.$$

Referring to the table on the last page, items 2 and 3 with $a = -3$ and $b = 2$ imply that

$$\mathcal{L}^{-1}\left[\frac{s+3}{(s+3)^2+2^2}\right] = e^{-3t}\cos(2t)\,, \qquad \mathcal{L}^{-1}\left[\frac{2}{(s+3)^2+2^2}\right] = e^{-3t}\sin(2t)\,.$$

The above formulas and the linearity of the inverse Laplace transform yield

$$\begin{aligned} j(t) &= \mathcal{L}^{-1}[J(s)](t) = \mathcal{L}^{-1}\left[\frac{3s+11}{s^2+6s+13}\right](t) \\ &= \mathcal{L}^{-1}\left[3\,\frac{s+3}{(s+3)^2+2^2} + \frac{2}{(s+3)^2+2^2}\right](t) \\ &= 3\mathcal{L}^{-1}\left[\frac{s+3}{(s+3)^2+2^2}\right](t) + \mathcal{L}^{-1}\left[\frac{2}{(s+3)^2+2^2}\right](t) \\ &= 3e^{-3t}\cos(2t) + e^{-3t}\sin(2t)\,. \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{L}^{-1}\big[Y(s)\big](t) &= \mathcal{L}^{-1}[e^{-4s}J(s)](t) = u(t-4)j(t-4) \\ &= u(t-4)\left(3e^{-3(t-4)}\cos(t-4) + e^{-3(t-4)}\sin(t-4)\right). \end{aligned}$$

Table of Laplace Transforms

$\mathcal{L}[t^n e^{at}](s) = \dfrac{n!}{(s-a)^{n+1}}$	for $s > a$.
$\mathcal{L}[e^{at}\cos(bt)](s) = \dfrac{s-a}{(s-a)^2+b^2}$	for $s > a$.
$\mathcal{L}[e^{at}\sin(bt)](s) = \dfrac{b}{(s-a)^2+b^2}$	for $s > a$.
$\mathcal{L}[t^n j(t)](s) = (-1)^n J^{(n)}(s)$	where $J(s) = \mathcal{L}[j(t)](s)$.
$\mathcal{L}[e^{at} j(t)](s) = J(s-a)$	where $J(s) = \mathcal{L}[j(t)](s)$.
$\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}J(s)$	where $J(s) = \mathcal{L}[j(t)](s)$ and u is the unit step function.
$\mathcal{L}[\delta(t-c)h(t)](s) = e^{-cs}h(c)$	where δ is the unit impluse.