Third In-Class Exam
Math 246, Professor David Levermore
Tuesday, 21 November 2017

Your Name: ______________________________

UMD SID: ______________________________

Discussion Instructor (circle one): Yan Tay Jing Zhou
Discussion Time (circle one): 8:00 9:00 10:00

No books, notes, calculators, or any electronic devices. If you need more space to answer a problem then use the back of one of these pages. Clearly indicate where your answer to each part of every problem is located. **Give your reasoning for full credit.** Any work that you do not want to be considered should be crossed out. Good luck!

University Honor Pledge: *I pledge on my honor that I have not given or received any unauthorized assistance on this examination.*

Signature: ______________________________

Problem 1:	________/6	Problem 2:	________/10
Problem 3:	________/6	Problem 4:	________/10
Problem 5:	________/6	Problem 6:	________/8
Problem 7:	________/8	Problem 8:	________/8
Problem 9:	________/10	Problem 10:	________/8
Problem 11:	________/12	Problem 12:	________/8
		Total Score:	________/100

Grade: ________

Name: ____________________________

(1) [6] Recast the ordinary differential equation $v'''' = \cos(v)v''' + (v'')^4 + \sin(t^2 + v')$ as a first-order system of ordinary differential equations.

(2) [10] Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} t^4 \\ 1 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} -e^t \\ e^t \end{pmatrix}$.

(a) [2] Compute the Wronskian $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t)$.

(b) [3] Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ wherever $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

(c) [2] Give a general solution to the system found in part (b).

(d) [3] Compute the Green matrix associated with the system found in part (b).

Name: ______________________________

(3) [6] Two interconnected tanks are filled with brine (salt water). At $t = 0$ the first tank contains 23 liters and the second contains 32 liters. Brine with a salt concentration of 8 grams per liter flows into the first tank at 6 liters per hour. Well-stirred brine flows from the first tank into the second at 7 liters per hour, from the second into the first at 5 liters per hour, from the first into a drain at 3 liter per hour, and from the second into a drain at 4 liters per hour. At $t = 0$ there are 17 grams of salt in the first tank and 29 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

(4) [10] Solve the initial-value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

Name: ______________________________

(5) [6] Given that 3 is an eigenvalue of the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 0 & -3 \\ 0 & 5 & 4 \\ 2 & 2 & 1 \end{pmatrix},$$

find all the eigenvectors of $\mathbf{A}$ associated with 3.

(6) [8] A 4×4 matrix $\mathbf{A}$ has the eigenpairs

$$\left(3, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right), \quad \left(4, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}\right), \quad \left(-1, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}\right), \quad \left(-2, \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}\right).$$

(a) Give an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ such that $e^{t\mathbf{A}} = \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1}$. (You do not have to compute either $\mathbf{V}^{-1}$ or $e^{t\mathbf{A}}$!)

(b) Give a fundamental matrix for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Name: ______________________________

(7) [8] Find a real general solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

(8) [8] Find a real general solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ -5 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Name: ____

(9) [10] Sketch the phase-plane portrait for each of the systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ from the previous two problems. Indicate typical orbits. Identify the type of this phase-plane portrait. Give a reason why the origin is either attracting, stable, unstable, or repelling.

(a) $\mathbf{A} = \begin{pmatrix} 0 & 2 \\ 5 & 3 \end{pmatrix}$, (b) $\mathbf{A} = \begin{pmatrix} -4 & 1 \\ -5 & -2 \end{pmatrix}$.

(10) [8] Compute the Laplace transform of $f(t) = u(t-2)\, e^{-4t}$ from its definition. (Here u is the unit step function.)

Name: ____________________________________

(11) [12] Consider the following MATLAB commands.

```
>> syms t s Y; f = ['t + heaviside(t - 3)*(6 - t) - heaviside(t - 6)*6'];
>> diffeqn = sym('D(D(y))(t) + 4*D(y)(t) + 20*y(t) = ' f);
>> eqntrans = laplace(diffeqn, t, s);
>> algeqn = subs(eqntrans, {'laplace(y(t),t,s),t,s)', 'y(0)', 'D(y)(0)'}, {Y, -2, 5});
>> ytrans = simplify(solve(algeqn, Y));
>> y = ilaplace(ytrans, s, t)
```

(a) [4] Give the initial-value problem for $y(t)$ that is being solved.

(b) [8] Find the Laplace transform $Y(s)$ of the solution $y(t)$. (DO NOT take the inverse Laplace transform of $Y(s)$ to find $y(t)$, just solve for $Y(s)$!)

You may refer to the table on the last page.

(12) [8] Find the inverse Laplace transform $\mathcal{L}^{-1}[Y(s)](t)$ of the function

$$Y(s) = e^{-3s}\,\frac{3s + 10}{s^2 + 4s + 5}\,.$$

You may refer to the table on the last page.

Name: ____

Workspace (Give the number of the problem being worked!)

A Short Table of Laplace Transforms

$$\mathcal{L}[t^n e^{at}](s) = \frac{n!}{(s-a)^{n+1}} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\cos(bt)](s) = \frac{s-a}{(s-a)^2+b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\sin(bt)](s) = \frac{b}{(s-a)^2+b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[t^n j(t)](s) = (-1)^n J^{(n)}(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[e^{at} j(t)](s) = J(s-a) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}J(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)$$

and u is the unit step function.