Third In-Class Exam Solutions
Math 246, Professor David Levermore
Tuesday, 21 November 2017

(1) [6] Recast the ordinary differential equation $v'''' = \cos(v)v''' + (v'')^4 + \sin(t^2 + v')$ as a first-order system of ordinary differential equations.

Solution. Because the equation is fourth order, the first-order system must have dimension at least four. The simplest such first-order system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ \cos(x_1)x_4 + (x_3)^4 + \sin(t^2 + x_2) \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} v \\ v' \\ v'' \\ v''' \end{pmatrix}.$$

(2) [10] Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} t^4 \\ 1 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} -e^t \\ e^t \end{pmatrix}$.

(a) [2] Compute the Wronskian $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t)$.

(b) [3] Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ wherever $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

(c) [2] Give a general solution to the system found in part (b).

(d) [3] Compute the Green matrix associated with the system found in part (b).

Solution (a). The Wronskian is

$$\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix} t^4 & -e^t \\ 1 & e^t \end{pmatrix} = t^4 \cdot e^t - 1 \cdot (-e^t) = (t^4 + 1)e^t \,.$$

Solution (b). Let $\boldsymbol{\Psi}(t) = \begin{pmatrix} t^4 & -e^t \\ 1 & e^t \end{pmatrix}$. Because $\boldsymbol{\Psi}'(t) = \mathbf{A}(t)\boldsymbol{\Psi}(t)$, we have

$$\begin{aligned} \mathbf{A}(t) = \boldsymbol{\Psi}'(t)\boldsymbol{\Psi}(t)^{-1} &= \begin{pmatrix} 4t^3 & -e^t \\ 0 & e^t \end{pmatrix}\begin{pmatrix} t^4 & -e^t \\ 1 & e^t \end{pmatrix}^{-1} \\ &= \frac{1}{(t^4+1)e^t}\begin{pmatrix} 4t^3 & -e^t \\ 0 & e^t \end{pmatrix}\begin{pmatrix} e^t & e^t \\ -1 & t^4 \end{pmatrix} \\ &= \frac{1}{(t^4+1)e^t}\begin{pmatrix} 4t^3e^t + e^t & 4t^3e^t - t^4e^t \\ -e^t & t^4e^t \end{pmatrix} = \frac{1}{t^4+1}\begin{pmatrix} 4t^3 + 1 & 4t^3 - t^4 \\ -1 & t^4 \end{pmatrix}. \end{aligned}$$

Solution (c). A general solution is

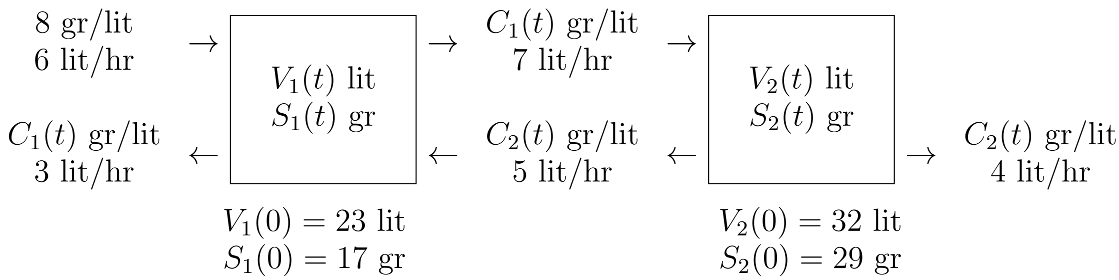
$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1\begin{pmatrix} t^4 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -e^t \\ e^t \end{pmatrix}.$$

Solution (d). By using the fundamental matrix $\boldsymbol{\Psi}(t)$ from part (b) we find that the Green matrix is

$$\begin{aligned} \mathbf{G}(t,s) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}(s)^{-1} &= \begin{pmatrix} t^4 & -e^t \\ 1 & e^t \end{pmatrix}\begin{pmatrix} s^4 & -e^s \\ 1 & e^s \end{pmatrix}^{-1} \\ &= \frac{1}{(s^4+1)e^s}\begin{pmatrix} t^4 & -e^t \\ 1 & e^t \end{pmatrix}\begin{pmatrix} e^s & e^s \\ -1 & s^4 \end{pmatrix} = \frac{1}{(s^4+1)e^s}\begin{pmatrix} t^4e^s + e^t & t^4e^s - e^ts^4 \\ e^s - e^t & e^s + e^ts^4 \end{pmatrix}. \end{aligned}$$

(3) [6] Two interconnected tanks are filled with brine (salt water). At $t = 0$ the first tank contains 23 liters and the second contains 32 liters. Brine with a salt concentration of 8 grams per liter flows into the first tank at 6 liters per hour. Well-stirred brine flows from the first tank into the second at 7 liters per hour, from the second into the first at 5 liters per hour, from the first into a drain at 3 liter per hour, and from the second into a drain at 4 liters per hour. At $t = 0$ there are 17 grams of salt in the first tank and 29 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

Solution. Let $V_1(t)$ and $V_2(t)$ be the volumes (lit) of brine in the first and second tank at time t hours. Let $S_1(t)$ and $S_2(t)$ be the mass (gr) of salt in the first and second tank at time t hours. Because the mixtures are assumed to be well-stirred, the salt concentration of the brine in the tanks at time t are $C_1(t) = S_1(t)/V_1(t)$ and $C_2(t) = S_2(t)/V_2(t)$ respectively. In particular, these are the concentrations of the brine that flows out of these tanks. We have the following picture.



We are asked to write down an initial-value problem that governs $S_1(t)$ and $S_2(t)$.

The rates work out so there will be $V_1(t) = 23 + t$ liters of brine in the first tank and $V_2(t) = 32 - 2t$ liters in the second. Then $S_1(t)$ and $S_2(t)$ are governed by the initial-value problem

$$\begin{aligned} \frac{\mathrm{d}S_1}{\mathrm{d}t} &= 8 \cdot 6 + \frac{S_2}{32-2t}5 - \frac{S_1}{23+t}7 - \frac{S_1}{23+t}3\,, \qquad & S_1(0) = 17\,, \\ \frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{S_1}{23+t}7 - \frac{S_2}{32-2t}5 - \frac{S_2}{32-2t}4\,, \qquad & S_2(0) = 29\,. \end{aligned}$$

Your answer could be left in the above form. However, it can be simplified to

$$\begin{aligned} \frac{\mathrm{d}S_1}{\mathrm{d}t} &= 48 + \frac{5}{32-2t}S_2 - \frac{10}{23+t}S_1\,, \qquad & S_1(0) = 17\,, \\ \frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{7}{23+t}S_1 - \frac{9}{32-2t}S_2\,, \qquad & S_2(0) = 29\,. \end{aligned}$$

Remark. This first-order system of differential equations is *linear*. Its coefficients are undefined either at $t = -23$ or at $t = 16$ and are continuous elsewhere. Its forcing is constant, so is continuous everywhere. Therefore the natural interval of definition for the solution of this initial-value problem is $(-23, 16)$ because:

- the initial time $t = 0$ is in $(-23, 16)$;
- all the coefficients and the forcing are continuous over $(-23, 16)$;
- two coefficients are undefined at $t = -23$;
- two coefficients are undefined at $t = 16$.

However, it could also be argued that the interval of definition for the solution of this initial-value problem is $[0, 16)$ because the word problem starts at $t = 0$.

(4) [10] Solve the initial-value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix}$ is

$$p(z) = z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 8z + 16 = (z-4)^2 .$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which is the double root 4. Then

$$e^{t\mathbf{A}} = e^{4t}\left[\mathbf{I} + t\,(\mathbf{A} - 4\mathbf{I})\right] = e^{4t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}\right] = e^{4t}\begin{pmatrix} 1-2t & -t \\ 4t & 1+2t \end{pmatrix}.$$

(Check that $\mathrm{tr}(\mathbf{A} - 4\mathbf{I}) = 0$!) Therefore the solution of the initial-value problem is

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}^I = e^{4t}\begin{pmatrix} 1-2t & -t \\ 4t & 1+2t \end{pmatrix}\begin{pmatrix} 0 \\ 3 \end{pmatrix} = e^{4t}\begin{pmatrix} -3t \\ 3+6t \end{pmatrix}.$$

(5) [6] Given that 3 is an eigenvalue of the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 0 & -3 \\ 0 & 5 & 4 \\ 2 & 2 & 1 \end{pmatrix},$$

find all the eigenvectors of $\mathbf{A}$ associated with 3.

Solution. The eigenvectors of $\mathbf{A}$ associated with 3 are all nonzero vectors $\mathbf{v}$ such that $\mathbf{A}\mathbf{v} = 3\mathbf{v}$. Equivalently, they are all nonzero vectors $\mathbf{v}$ such that $(\mathbf{A} - 3\mathbf{I})\mathbf{v} = \mathbf{0}$, which is

$$\begin{pmatrix} 1 & 0 & -3 \\ 0 & 2 & 4 \\ 2 & 2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The entries of $\mathbf{v}$ thereby satisfy the homogeneous linear algebraic system

$$\begin{aligned} v_1 \qquad\quad - 3v_3 &= 0\,, \\ 2v_2 + 4v_3 &= 0\,, \\ 2v_1 + 2v_2 - 2v_3 &= 0\,. \end{aligned}$$

This system may be solved either by elimination or by row reduction. By either method its general solution is found to be

$$v_1 = 3\alpha\,, \quad v_2 = -2\alpha\,, \quad v_3 = \alpha\,, \qquad \text{for any constant } \alpha\,.$$

Therefore the eigenvectors of $\mathbf{A}$ associated with 3 each have the form

$$\alpha\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \qquad \text{for some constant } \alpha \neq 0\,.$$

(6) [8] A 4×4 matrix $\mathbf{A}$ has the eigenpairs

$$\left(3\,, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right), \qquad \left(4\,, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}\right), \qquad \left(-1\,, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}\right), \qquad \left(-2\,, \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}\right).$$

(a) Give an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ such that $e^{t\mathbf{A}} = \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1}$. (You do not have to compute either $\mathbf{V}^{-1}$ or $e^{t\mathbf{A}}$!)

(b) Give a fundamental matrix for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Solution (a). One choice for $\mathbf{V}$ and $\mathbf{D}$ is

$$\mathbf{V} = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & -1 \\ 1 & 0 & -1 & 1 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

Solution (b). Use the given eigenpairs to construct the real eigensolutions

$$\mathbf{x}_1(t) = e^{3t}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{x}_2(t) = e^{4t}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix},$$

$$\mathbf{x}_3(t) = e^{-t}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \mathbf{x}_4(t) = e^{-2t}\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

Then a fundamental matrix for the system is

$$\mathbf{\Psi}(t) = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \mathbf{x}_3(t) & \mathbf{x}_4(t) \end{pmatrix} = \begin{pmatrix} e^{3t} & e^{4t} & 0 & -e^{-2t} \\ e^{3t} & 0 & e^{-t} & e^{-2t} \\ e^{3t} & -e^{4t} & 0 & -e^{-2t} \\ e^{3t} & 0 & -e^{-t} & e^{-2t} \end{pmatrix}.$$

Alternative Solution (b). Given the $\mathbf{V}$ and $\mathbf{D}$ from part (a), a fundamental matrix for the system is

$$\begin{aligned} \mathbf{\Psi}(t) = \mathbf{V}e^{t\mathbf{D}} &= \begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & -1 \\ 1 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 & 0 & 0 \\ 0 & e^{4t} & 0 & 0 \\ 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & e^{-2t} \end{pmatrix} \\ &= \begin{pmatrix} e^{3t} & e^{4t} & 0 & -e^{-2t} \\ e^{3t} & 0 & e^{-t} & e^{-2t} \\ e^{3t} & -e^{4t} & 0 & -e^{-2t} \\ e^{3t} & 0 & -e^{-t} & e^{-2t} \end{pmatrix}. \end{aligned}$$

(7) [8] Find a real general solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 0 & 2 \\ 5 & 3 \end{pmatrix}$ is

$$p(z) = z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 3z - 10 = (z-5)(z+2).$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are -2 and 5. Consider the matrices

$$\mathbf{A} + 2\mathbf{I} = \begin{pmatrix} 2 & 2 \\ 5 & 5 \end{pmatrix}, \qquad \mathbf{A} - 5\mathbf{I} = \begin{pmatrix} -5 & 2 \\ 5 & -2 \end{pmatrix}.$$

After checking that the determinant of each matrix is zero, we can read off from their first columns that eigenpairs of $\mathbf{A}$ are

$$\left(-2, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right), \qquad \left(5, \begin{pmatrix} 2 \\ 5 \end{pmatrix}\right).$$

Therefore a real general solution of the system is

$$\mathbf{x}(t) = c_1 e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{5t}\begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

Alternative Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 0 & 2 \\ 5 & 3 \end{pmatrix}$ is

$$p(z) = z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 3z - 10 = (z-5)(z+2).$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are -2 and 5, which are $\frac{3}{2} - \frac{7}{2}$ and $\frac{3}{2} + \frac{7}{2}$. Then

$$\begin{aligned}
e^{t\mathbf{A}} &= e^{\frac{3}{2}t}\left[\cosh(\tfrac{7}{2}t)\mathbf{I} + \frac{\sinh(\frac{7}{2}t)}{\frac{7}{2}}\left(\mathbf{A} - \tfrac{3}{2}\mathbf{I}\right)\right] \\
&= e^{\frac{3}{2}t}\left[\cosh(\tfrac{7}{2}t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sinh(\frac{7}{2}t)}{\frac{7}{2}}\begin{pmatrix} -\frac{3}{2} & 2 \\ 5 & \frac{3}{2} \end{pmatrix}\right] \\
&= e^{\frac{3}{2}t}\begin{pmatrix} \cosh(\frac{7}{2}t) - \frac{3}{7}\sinh(\frac{7}{2}t) & \frac{4}{7}\sinh(\frac{7}{2}t) \\ \frac{10}{7}\sinh(\frac{7}{2}t) & \cosh(\frac{7}{2}t) + \frac{3}{7}\sinh(\frac{7}{2}t) \end{pmatrix}.
\end{aligned}$$

(Check that $\mathrm{tr}(\mathbf{A} - \frac{3}{2}\mathbf{I}) = 0$!) Therefore a real general solution of the system is

$$\begin{aligned}
\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{c} &= e^{\frac{3}{2}t}\begin{pmatrix} \cosh(\frac{7}{2}t) - \frac{3}{7}\sinh(\frac{7}{2}t) & \frac{4}{7}\sinh(\frac{7}{2}t) \\ \frac{10}{7}\sinh(\frac{7}{2}t) & \cosh(\frac{7}{2}t) + \frac{3}{7}\sinh(\frac{7}{2}t) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\
&= c_1 e^{\frac{3}{2}t}\begin{pmatrix} \cosh(\frac{7}{2}t) - \frac{3}{7}\sinh(\frac{7}{2}t) \\ \frac{10}{7}\sinh(\frac{7}{2}t) \end{pmatrix} + c_2 e^{\frac{3}{2}t}\begin{pmatrix} \frac{4}{7}\sinh(\frac{7}{2}t) \\ \cosh(\frac{7}{2}t) + \frac{3}{7}\sinh(\frac{7}{2}t) \end{pmatrix}.
\end{aligned}$$

(8) [8] Find a real general solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ -5 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} -4 & 1 \\ -5 & -2 \end{pmatrix}$ is

$$p(z) = z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 6z + 13 = (z-3)^2 + 2^2.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $-3+i2$ and $-3-i2$. Then

$$\begin{aligned} e^{t\mathbf{A}} &= e^{-3t}\left[\cos(2t)\mathbf{I} + \frac{\sin(2t)}{2}(\mathbf{A}+3\mathbf{I})\right] \\ &= e^{-3t}\left[\cos(2t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(2t)}{2}\begin{pmatrix} -1 & 1 \\ -5 & 1 \end{pmatrix}\right] \\ &= e^{-3t}\begin{pmatrix} \cos(2t) - \frac{1}{2}\sin(2t) & \frac{1}{2}\sin(2t) \\ -\frac{5}{2}\sin(2t) & \cos(2t) + \frac{1}{2}\sin(2t) \end{pmatrix}. \end{aligned}$$

(Check that $\mathrm{tr}(\mathbf{A}+3\mathbf{I}) = 0$!) Therefore a real general solution of the system is

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{c} = c_1 e^{-3t}\begin{pmatrix} \cos(2t) - \frac{1}{2}\sin(2t) \\ -\frac{5}{2}\sin(2t) \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} \frac{1}{2}\sin(2t) \\ \cos(2t) + \frac{1}{2}\sin(2t) \end{pmatrix}.$$

Alternative Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} -4 & 1 \\ -5 & -2 \end{pmatrix}$ is

$$p(z) = z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 6z + 13 = (z-3)^2 + 2^2.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $-3+i2$ and $-3-i2$. Consider the matrix

$$\mathbf{A} - (-3-i2)\mathbf{I} = \begin{pmatrix} -1+i2 & 1 \\ -5 & 1+i2 \end{pmatrix}.$$

After checking that the determinant of this matrix is zero, we can read off from its second column that an eigenpair of $\mathbf{A}$ is

$$\left(-3+i2\,, \begin{pmatrix} 1 \\ 1+i2 \end{pmatrix}\right).$$

(Another eigenpair is the complex conjugate of this one, but we will not need it.) This eigenpair yields the complex-valued eigensolution

$$\begin{aligned} \mathbf{x}(t) &= e^{(-3+i2)t}\begin{pmatrix} 1 \\ 1+i2 \end{pmatrix} = e^{-3t}\big(\cos(2t) + i\sin(2t)\big)\begin{pmatrix} 1 \\ 1+i2 \end{pmatrix} \\ &= e^{-3t}\begin{pmatrix} \cos(2t) + i\sin(2t) \\ \big(\cos(2t) + i\sin(2t)\big)(1+i2) \end{pmatrix} \\ &= e^{-3t}\begin{pmatrix} \cos(2t) + i\sin(2t) \\ \big(\cos(2t) - 2\sin(2t)\big) + i\big(2\cos(2t) + \sin(2t)\big) \end{pmatrix}. \end{aligned}$$

From the real and imaginary parts of this complex-valued eigensolution we can read off that a fundamental set of real solutions is

$$\mathbf{x}_1(t) = e^{-3t}\begin{pmatrix}\cos(2t)\\ \cos(2t) - 2\sin(2t)\end{pmatrix}, \qquad \mathbf{x}_2(t) = e^{-3t}\begin{pmatrix}\sin(2t)\\ 2\cos(2t) + \sin(2t)\end{pmatrix}.$$

Therefore a real general solution is

$$\mathbf{x}(t) = c_1 e^{-3t}\begin{pmatrix}\cos(2t)\\ \cos(2t) - 2\sin(2t)\end{pmatrix} + c_2 e^{-3t}\begin{pmatrix}\sin(2t)\\ 2\cos(2t) + \sin(2t)\end{pmatrix}.$$

(9) [10] Sketch the phase-plane portrait for each of the systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ from the previous two problems. Indicate typical orbits. Identify the type of this phase-plane portrait. Give a reason why the origin is either attracting, stable, unstable, or repelling.

$$\text{(a) } \mathbf{A} = \begin{pmatrix}0 & 2\\ 5 & 3\end{pmatrix}, \qquad \text{(b) } \mathbf{A} = \begin{pmatrix}-4 & 1\\ -5 & -2\end{pmatrix}.$$

Solution (a). Because the characteristic polynomial of $\mathbf{A}$ is $p(z) = (z+2)(z-5)$, the eigenvalues of $\mathbf{A}$ are -2 and 5. Because these eigenvalues are real and have opposite sign, the phase-plane portrait is a *saddle*. Therefore the origin is *unstable*, but not repelling. There are real eigenpairs (see the solution to Problem 7 for details)

$$\left(-2, \begin{pmatrix}1\\ -1\end{pmatrix}\right), \qquad \left(5, \begin{pmatrix}2\\ 5\end{pmatrix}\right).$$

Therefore you should sketch eigensolution orbits that approach the origin along the line $y = -x$ and eigensolution orbits that emerge from the origin along the line $y = \frac{5}{2}x$. You should sketch one representative orbit in each of the four regions separated by the eigensolution orbits. Each of these four orbits asymptotes to the line $y = -x$ as $t \to -\infty$ and asymptotes to the line $y = \frac{5}{2}x$ as $t \to \infty$.

Solution (b). Because the characteristic polynomial of $\mathbf{A}$ is $p(z) = (z+3)^2 + 4$, the eigenvalues of $\mathbf{A}$ are $-3 + i2$ and $-3 - i2$. Because these eigenvalues are a conjugate pair with negative real part, the phase-plane portrait is a spiral sink. Because $a_{21} = -5 < 0$, the right-hand rule says that it is a *clockwise spiral sink*. Therefore the origin is *attracting*. There are no real eigenpairs, so there are no eigensolution orbits to sketch. The phase portrait should indicate a family of clockwise spiral orbits that approach the origin.

(10) [8] Compute the Laplace transform of $f(t) = u(t-2)\, e^{-4t}$ from its definition. (Here u is the unit step function.)

Solution. The definition of Laplace transform gives

$$\mathcal{L}[f](s) = \lim_{T\to\infty} \int_0^T e^{-st} u(t-2)\, e^{-4t}\, \mathrm{d}t = \lim_{T\to\infty} \int_2^T e^{-(s+4)t}\, \mathrm{d}t.$$

When $s \le -4$ this limit diverges to $+\infty$ because in that case we have for every $T > 2$

$$\int_2^T e^{-(s+4)t}\, \mathrm{d}t \ge \int_2^T \mathrm{d}t = T - 2,$$

which clearly diverges to $+\infty$ as $T \to \infty$.

When $s > -4$ we have for every $T > 2$

$$\int_2^T e^{-(s+4)t}\,\mathrm{d}t = -\frac{e^{-(s+4)t}}{s+4}\bigg|_2^T = -\frac{e^{-(s+4)T}}{s+4} + \frac{e^{-(s+4)2}}{s+4}\,,$$

whereby

$$\mathcal{L}[f](s) = \lim_{T\to\infty}\left[-\frac{e^{-(s+4)T}}{s+4} + \frac{e^{-(s+4)2}}{s+4}\right] = \frac{e^{-(s+4)2}}{s+4} \qquad \text{for } s > -4\,.$$

Therefore the definition of the Laplace transform shows that

$$\mathcal{L}[f](s) = \begin{cases} \dfrac{e^{-(s+4)2}}{s+4} & \text{for } s > -4\,,\\ \text{undefined} & \text{for } s \leq -4\,.\end{cases}$$

(11) [12] Consider the following MATLAB commands.

```
>> syms t s Y; f = ['t + heaviside(t − 3)*(6 − t) − heaviside(t − 6)*6'];
>> diffeqn = sym('D(D(y))(t) + 4*D(y)(t) + 20*y(t) = ' f);
>> eqntrans = laplace(diffeqn, t, s);
>> algeqn = subs(eqntrans, {'laplace(y(t),t,s),t,s)', 'y(0)', 'D(y)(0)'}, {Y, −2, 5});
>> ytrans = simplify(solve(algeqn, Y));
>> y = ilaplace(ytrans, s, t)
```

(a) [4] Give the initial-value problem for $y(t)$ that is being solved.
(b) [8] Find the Laplace transform $Y(s)$ of the solution $y(t)$. (DO NOT take the inverse Laplace transform of $Y(s)$ to find $y(t)$, just solve for $Y(s)$!)

You may refer to the table on the last page.

Solution (a). The initial-value problem for $y(t)$ that is being solved is

$$y'' + 4y' + 20y = f(t)\,, \qquad y(0) = -2\,, \quad y'(0) = 5\,,$$

where the forcing $f(t)$ can be expressed either as the piecewise-defined function

$$f(t) = \begin{cases} t & \text{for } 0 \leq t < 3\,,\\ 6 & \text{for } 3 \leq t < 6\,,\\ 0 & \text{for } 6 \leq t\,,\end{cases}$$

or in terms of the unit step function as $f(t) = t + u(t-3)(6-t) - u(t-6)6$.

Solution (b). The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y'](s) + 20\mathcal{L}[y](s) = \mathcal{L}[f](s)\,.$$

Because

$$\begin{aligned}\mathcal{L}[y](s) &= Y(s)\,,\\ \mathcal{L}[y'](s) &= s\,\mathcal{L}[y](s) - y(0) = sY(s) + 2\,,\\ \mathcal{L}[y''](s) &= s\,\mathcal{L}[y'](s) - y'(0) = s^2Y(s) + 2s - 5\,,\end{aligned}$$

the Laplace transform of the initial-value problem becomes

$$\big(s^2Y(s) + 2s - 5\big) + 4\big(sY(s) + 2\big) + 20Y(s) = \mathcal{L}[f](s)\,.$$

This simplifies to

$$(s^2 + 4s + 20)Y(s) + 2s + 3 = \mathcal{L}[f](s)\,,$$

whereby

$$Y(s) = \frac{1}{s^2 + 4s + 20}\big(- 2s - 3 + \mathcal{L}[f](s)\big)\,.$$

To compute $\mathcal{L}[f](s)$, we write $f(t)$ as

$$\begin{aligned} f(t) &= t + u(t-3)(6-t) - u(t-6)6 \\ &= t + u(t-3)j_1(t-3) + u(t-6)j_2(t-6)\,, \end{aligned}$$

where by setting $j_1(t-3) = 6 - t$ and $j_2(t-6) = -6$ we see that

$$j_1(t) = 6 - (t+3) = 3 - t\,, \qquad j_2(t) = -6\,.$$

Referring to the table on the last page, item 1 with $a = 0$ and $n = 0$ and with $a = 0$ and $n = 1$ shows that

$$\mathcal{L}[1](s) = \frac{1}{s}\,, \qquad \mathcal{L}[t](s) = \frac{1}{s^2}\,,$$

whereby item 6 with $c = 3$ and $j(t) = j_1(t)$ and with $c = 3$ and $j(t) = j_2(t)$ shows that

$$\begin{aligned} \mathcal{L}\big[u(t-3)j_1(t-3)\big](s) &= e^{-3s}\mathcal{L}[j_1](s) = e^{-3s}\mathcal{L}[3-t](s) = e^{-3s}\left(\frac{3}{s} - \frac{1}{s^2}\right)\,, \\ \mathcal{L}\big[u(t-6)j_2(t-6)\big](s) &= e^{-6s}\mathcal{L}[j_2](s) = -e^{-6s}\mathcal{L}[6](s) = -e^{-6s}\frac{6}{s}\,. \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{L}[f](s) &= \mathcal{L}\big[t + u(t-3)j_1(t-3) + u(t-6)j_2(t-6)\big](s) \\ &= \frac{1}{s^2} + e^{-3s}\left(\frac{3}{s} - \frac{1}{s^2}\right) - e^{-6s}\frac{6}{s}\,. \end{aligned}$$

Upon placing this result into the expression for $Y(s)$ found earlier, we obtain

$$Y(s) = \frac{1}{s^2 + 4s + 20}\left(- 2s - 3 + \frac{1}{s^2} + e^{-3s}\left(\frac{3}{s} - \frac{1}{s^2}\right) - e^{-6s}\frac{6}{s}\right)\,.$$

(12) [8] Find the inverse Laplace transform $\mathcal{L}^{-1}[Y(s)](t)$ of the function

$$Y(s) = e^{-3s}\,\frac{3s + 10}{s^2 + 4s + 5}\,.$$

You may refer to the table on the last page.

Solution. Referring to the table on the last page, item 6 with $c = 3$ implies that

$$\mathcal{L}^{-1}\big[e^{-3s}\,J(s)\big] = u(t-3)j(t-3)\,, \qquad \text{where} \qquad j(t) = \mathcal{L}^{-1}[J(s)](t)\,.$$

We apply this formula to

$$J(s) = \frac{3s + 10}{s^2 + 4s + 5}\,.$$

Because $s^2 + 4s + 5 = (s+2)^2 + 1$, we have the partial fraction identity

$$J(s) = \frac{3s+10}{s^2+4s+5} = \frac{3(s+2)+4}{(s+2)^2+1} = 3\,\frac{s+2}{(s+2)^2+1} + 4\,\frac{1}{(s+2)^2+1}\,.$$

Referring to the table on the last page, items 2 and 3 with $a = -2$ and $b = 1$ imply that

$$\mathcal{L}^{-1}\left[\frac{s+2}{(s+2)^2+1}\right] = e^{-2t}\cos(t)\,, \qquad \mathcal{L}^{-1}\left[\frac{1}{(s+2)^2+1}\right] = e^{-2t}\sin(t)\,.$$

The above formulas and the linearity of the inverse Laplace transform yield

$$\begin{aligned} j(t) &= \mathcal{L}^{-1}[J(s)](t) = \mathcal{L}^{-1}\left[\frac{3s+10}{s^2+4s+5}\right](t) \\ &= \mathcal{L}^{-1}\left[3\,\frac{s+2}{(s+2)^2+1} + 4\,\frac{1}{(s+2)^2+1}\right](t) \\ &= 3\mathcal{L}^{-1}\left[\frac{s+2}{(s+2)^2+1}\right](t) + 4\mathcal{L}^{-1}\left[\frac{1}{(s+2)^2+1}\right](t) \\ &= 3e^{-2t}\cos(t) + 4e^{-2t}\sin(t)\,. \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{L}^{-1}\big[Y(s)\big](t) &= \mathcal{L}^{-1}[e^{-3s}J(s)](t) = u(t-3)j(t-3) \\ &= u(t-3)\left(3e^{-2(t-3)}\cos(t-3) + 4e^{-2(t-3)}\sin(t-3)\right). \end{aligned}$$

A Short Table of Laplace Transforms

$$\mathcal{L}[t^n e^{at}](s) = \frac{n!}{(s-a)^{n+1}} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\cos(bt)](s) = \frac{s-a}{(s-a)^2+b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\sin(bt)](s) = \frac{b}{(s-a)^2+b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[t^n j(t)](s) = (-1)^n J^{(n)}(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[e^{at} j(t)](s) = J(s-a) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}J(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)$$

and u is the unit step function.