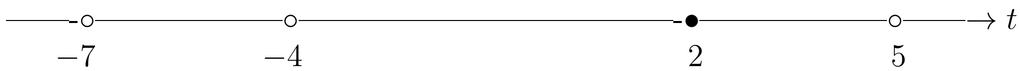
Second In-Class Exam Solutions
Math 246, Professor David Levermore
Tuesday, 19 October 2017

(1) [4] Give the interval of definition for the solution of the initial-value problem

$$u''' - \frac{\sin(3t)}{4+t}\,u'' + \frac{5+t}{5-t}\,u = \frac{e^{-t}}{7+t}\,, \qquad u(2) = u'(2) = u''(2) = -3\,.$$

Solution. The equation is linear and is already in normal form. Notice the following.
- ⋄ The coefficient of u'' is undefined at $t = -4$ and is continuous elsewhere.
- ⋄ The coefficient of u is undefined at $t = 5$ and is continuous elsewhere.
- ⋄ The forcing is undefined at $t = -7$ and is continuous elsewhere.

Plotting these points along with the inital time $t = 2$ on a time-line gives



Therefore the interval of definition is $(-4, 5)$ because:
- the initial time $t = 2$ is in $(-4, 5)$;
- all the coefficients and the forcing are continuous over $(-4, 5)$;
- the coefficient of u'' is undefined at $t = -4$;
- the coefficient of u is undefined at $t = 5$.

Remark. All four reasons must be given for full credit.
- ∘ The first two reasons are why a (unique) solution exists over the interval $(-4, 5)$.
- ∘ The last two reasons are why this solution does not exist over a larger interval.

(2) [12] Let L be a linear ordinary differential operator with constant coefficients. Suppose that all the roots of its characteristic polynomial (listed with their multiplicities) are $-2+i5$, $-2+i5$, $-2-i5$, $-2-i5$, $i7$, $-i7$, -3, -3, 4, 0, 0, 0.

(a) [2] Give the order of L.

(b) [10] Give a real general solution of the homogeneous equation $\mathrm{L}y = 0$.

Solution (a). Because there are 12 roots listed, the degree of the characteristic polynomial must be 12, whereby the order of L is 12.

Solution (b). A fundamental set of twelve real-valued solutions is built as follows.
- ⋄ The conjugate pair of double roots $-2 \pm i5$ contributes
$$e^{-2t}\cos(5t)\,, \quad e^{-2t}\sin(5t)\,, \quad t\,e^{-2t}\cos(5t)\,, \quad \text{and} \quad t\,e^{-2t}\sin(3t)\,.$$
- ⋄ The conjugate pair of simple roots $\pm i7$ contributes
$$\cos(7t)\,, \quad \text{and} \quad \sin(7t)\,.$$
- ⋄ The double real root -3 contributes e^{-3t} and $t\,e^{-3t}$.
- ⋄ The simple real root 4 contributes e^{4t}.
- ⋄ The triple real root 0 contributes 1, t, and t^2.

Therefore a real general solution of the homogeneous equation $\mathrm{L}y = 0$ is

$$\begin{aligned} y = {} & c_1 e^{-2t}\cos(5t) + c_2 e^{-2t}\sin(5t) + c_3 t\,e^{-2t}\cos(5t) + c_4 t\,e^{-2t}\sin(5t) \\ & + c_5\cos(7t) + c_6\sin(7t) + c_7 e^{-3t} + c_8 t\,e^{-3t} + c_9 e^{4t} + c_{10} + c_{11}t + c_{12}t^2\,. \end{aligned}$$

(3) [4] Suppose that $V_1(t)$, $V_2(t)$, and $V_3(t)$ are solutions of the differential equation

$$v''' - 2v'' - \cos(4t)v' + (1+t^2)v = 0\,,$$

Suppose we know that $\mathrm{Wr}[V_1, V_2, V_3](0) = 3$. Find $\mathrm{Wr}[V_1, V_2, V_3](t)$.

Solution. The Abel Theorem says that $w(t) = \mathrm{Wr}[V_1, V_2, V_3](t)$ satisfies $w' - 2w = 0$. It follows that $w(t) = ce^{2t}$ for some c. Because $w(0) = \mathrm{Wr}[V_1, V_2, V_3](0) = 3$, this initial condition implies that $w(0) = ce^{2\cdot 0} = 3$, whereby $c = 3$. Therefore $w(t) = 3e^{2t}$, which means that

$$\mathrm{Wr}[V_1, V_2, V_3](t) = 3e^{2t}\,.$$

(4) [12] The functions $\cos(4t)$ and $\sin(4t)$ are a fundamental set of solutions to $\ddot{x} + 16x = 0$.
(a) [9] Solve the general initial-value problem

$$\ddot{x} + 16x = 0\,, \qquad x(0) = x_0\,, \quad \dot{x}(0) = x_1\,.$$

(b) [3] Find the associated natural fundamental set of solutions to $\ddot{x} + 16x = 0$.

Solution (a). Because we are told that $\cos(4t)$ and $\sin(4t)$ constitute a fundamental set of solutions to $\ddot{x} + 16x = 0$, we know that a general solution is

$$x(t) = c_1\cos(4t) + c_2\sin(4t)\,.$$

Because

$$\dot{x}(t) = -4c_1\sin(4t) + 4c_2\cos(4t)\,,$$

when the general initial conditions are imposed, we find that

$$x(0) = c_1 = x_0\,, \qquad \dot{x}(0) = 4c_2 = x_1\,.$$

These relations imply that $c_1 = x_0$ and $c_2 = \frac{1}{4}x_1$. Therefore the solution of the general initial-value problem is

$$x(t) = x_0\cos(4t) + x_1\tfrac{1}{4}\sin(4t)\,.$$

Solution (b). The natural fundamental set of solutions associated with $t = 0$ is read off as the functions multiplying x_0 and x_1 in the solution of the general initial-value problem. These are

$$N_0(t) = \cos(4t)\,, \qquad N_1(t) = \tfrac{1}{4}\sin(4t)\,.$$

(5) [8] What answer will be produced by the following Matlab commands?

```
>> ode = 'D2y - 6*Dy + 18*y = 18*exp(3*t)';
>> dsolve(ode, 't')

ans =
```

Solution. The commands ask Matlab for a real general solution of the equation

$$\mathrm{D}^2y - 6\mathrm{D}y + 18y = 18e^{3t}\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

While your answer did not have to be given in Matlab format, Matlab will produce something equivalent to

```
2*exp(3*t) + C1*exp(3*t)*cos(3*t) + C2*exp(3*t)*sin(3*t)
```

This can be seen as follows. This is a *nonhomogeneous* linear equation for $y(t)$ with *constant* coefficients. The characteristic polynomial is

$$p(z) = z^2 - 6z + 18 = (z-3)^2 + 9 = (z-3)^2 + 3^2 \,.$$

It has the conjugate pair of roots $3 \pm i3$. A real general solution of the associated homogeneous problem is

$$y_H(t) = c_1 e^{3t}\cos(3t) + c_2 e^{3t}\sin(3t) \,.$$

The forcing $18e^{3t}$ has degree $d = 0$, characteristic $\mu + i\nu = 3$, and multiplicity $m = 0$. A particular solution $y_P(t)$ can be found by using either Key Identity Evaluations, the Zero Degree Formula, or Undetermined Coefficients. Below we show that each of these methods gives the particular solution $y_P(t) = 2e^{3t}$. Therefore a real general solution is

$$y = c_1 e^{3t}\cos(3t) + c_2 e^{3t}\sin(2t) + 2e^{3t} \,.$$

Up to notational differences, this is the answer that Matlab produces.

Key Indentity Evaluations. Because $m + d = m = 0$, we can simply evaluate the Key Identity at $z = \mu + i\nu = 3$, to find

$$\mathrm{L}(e^{3t}) = p(3)e^{3t} = (3^2 - 6\cdot 3 + 18)e^{3t} = 9e^{3t} \,.$$

Multiply this by 2 to obtain $\mathrm{L}(2e^{3t}) = 18e^{3t}$. Hence, $y_P(t) = 2e^{3t}$.

Zero Degree Formula. For a forcing $f(t)$ with degree $d = 0$, characteristic $\mu + i\nu$, and multiplicity m that has the form

$$f(t) = \alpha e^{\mu t}\cos(\nu t) + \beta e^{\mu t}\sin(\nu t) = e^{\mu t}\,\mathrm{Re}\big((\alpha - i\beta)e^{i\nu t}\big) \,,$$

this formula gives the particular solution

$$y_P(t) = t^m e^{\mu t}\,\mathrm{Re}\left(\frac{\alpha - i\beta}{p^{(m)}(\mu + i\nu)}\, e^{i\nu t}\right) \,.$$

For this problem $f(t) = 18e^{3t}$ and $p(z) = z^2 - 6z + 18$, so that $\mu + i\nu = 3$, $\alpha - i\beta = 18$, and $m = 0$, whereby

$$y_P(t) = e^{3t}\,\frac{18}{p(3)} = \frac{18}{3^2 - 6\cdot 3 + 18}\, e^{3t} = \frac{18}{9}\, e^{3t} = 2e^{3t} \,.$$

Undetermined Coefficients. Because $m + d = m = 0$ and $\mu + i\nu = 3$, there is a particular solution in the form

$$y_P(t) = Ae^{3t} \,.$$

Because

$$y_P'(t) = 3Ae^{3t} \,, \qquad y_P''(t) = 9Ae^{3t} \,,$$

we see that

$$\mathrm{L}y_P(t) = y_P''(t) - 6y_P'(t) + 18y_P(t) = [9Ae^{3t}] - 6[3Ae^{3t}] + 18[Ae^{3t}] = 9Ae^{3t} \,.$$

Setting $\mathrm{L}y_P(t) = 9Ae^{3t} = 18e^{3t}$, we see that $A = 2$. Hence, $y_P(t) = 2e^{3t}$.

(6) [8] Compute the Green function $g(t)$ associated with the differential operator

$$\mathrm{D}^2 + 6\mathrm{D} + 10\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

Solution. The Green function $g(t)$ satisfies

$$\mathrm{D}^2 g + 6\mathrm{D}g + 10g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 1\,.$$

The characteristic polynomial is

$$p(z) = z^2 + 6z + 10 = (z+3)^2 + 1 = (z+3)^2 + 1^2\,,$$

which has the conjugate pair of roots $-3 \pm i$. Hence, the general solution of the equation is

$$g(t) = c_1 e^{-3t}\cos(t) + c_2 e^{-3t}\sin(t)\,.$$

The first initial condition implies $0 = g(0) = c_!$, whereby

$$g(t) = c_2 e^{-3t}\sin(t)\,.$$

Because

$$g'(t) = c_2 e^{-3t}\cos(t) - 3c_2 e^{-3t}\sin(t)\,,$$

the second initial condition implies $1 = g'(0) = c_2$. Therefore the Green function associated with the differential operator is

$$g(t) = e^{-3t}\sin(t)\,.$$

(7) [8] Solve the initial-value problem

$$q'' + 6q' + 10q = \frac{4e^{-3t}}{\cos(t)}\,, \qquad q(0) = q'(0) = 0\,.$$

Solution. This is a nonhomogeneous linear equation with constant coefficients. Because its forcing does not have characteristic form, we cannot use either Key Identity Evaluations or Undetermined Coefficients. We will use the Green Function method. By the last problem the Green function for this problem is $g(t) = e^{-3t}\sin(t)$. Because the equation is in normal form, the initial time is 0, and both the initial values are 0, the solution to this inital-value problem is given by the Green function formula

$$\begin{aligned} q(t) = \int_0^t g(t-s)f(s)\,\mathrm{d}s &= \int_0^t e^{-3(t-s)}\sin(t-s)\,\frac{4e^{-3s}}{\cos(s)}\,\mathrm{d}s \\ &= 4e^{-3t}\int_0^t \frac{\sin(t-s)}{\cos(s)}\,\mathrm{d}s\,. \end{aligned}$$

By using the trig identity

$$\sin(t-s) = \sin(t)\cos(s) - \cos(t)\sin(s)\,,$$

we obtain

$$\begin{aligned} q(t) &= 4e^{-3t}\int_0^t \frac{\sin(t)\cos(s) - \cos(t)\sin(s)}{\cos(s)}\,\mathrm{d}s \\ &= 4e^{-3t}\sin(t)\int_0^t \mathrm{d}s - 4e^{-3t}\cos(t)\int_0^t \frac{\sin(s)}{\cos(s)}\,\mathrm{d}s \\ &= 4e^{-3t}\sin(t)\,t + 4e^{-3t}\cos(t)\,\log(|\cos(t)|)\,. \end{aligned}$$

Remark. The interval of definition for this initial-value problem is $(-\frac{\pi}{2}, \frac{\pi}{2})$. Over this interval $\cos(t)$ is positive. Therefore we could have written

$$q(t) = 4e^{-3t}\sin(t)\,t + 4e^{-3t}\cos(t)\,\log(\cos(t))\,.$$

Remark. This problem can also be solved by the general Green function method. However that approach is not as efficient because it does not use the fact the Green function $g(t)$ was already computed in the solution of the preceeding problem.

Remark. This problem can also be solved by using variation of parameters. However that approach is not as efficient because it does not directly solve the initial-value problem. Rather, after finding a particular solution the constants c_1 and c_2 in $q_H(t)$ must be determined to satisfy the initial conditions.

(8) [8] Find a particular solution $u_P(t)$ of the equation $u'' - u = 8e^t$.

Solution. This is a nonhomogeneous linear equation with constant coefficients. Its characteristic polynomial is

$$p(z) = z^2 - 1 = (z+1)(z-1)\,,$$

which has two simple real roots -1 and 1. The forcing $8e^t$ has characteristic form with degree $d = 0$ and characteristic $\mu + i\nu = 1$, which has multiplicity $m = 1$. Therefore we can use either Key Identity Evaluations, the Zero Degreee Formula, or Undetermined Coefficients to find a particular solution $u_P(t)$.

Key Indentity Evaluations. Because $m + d = m = 1$ we need just the first derivative with respect to z of the Key Identity. The Key Identity and its first derivative with respect to z are

$$\mathrm{L}\big(e^{zt}\big) = (z^2-1)e^{zt}\,, \qquad \mathrm{L}\big(t\,e^{zt}\big) = (z^2-1)t\,e^{zt} + 2ze^{zt}\,.$$

By evaluating the first derivative of the Key Identity at $z = \mu + i\nu = 1$ we find that

$$\mathrm{L}\big(t\,e^t\big) = 2e^t\,.$$

After multiplying this equation by 4 it becomes

$$\mathrm{L}\big(4t\,e^t\big) = 8e^t\,.$$

Therefore a particular solution of $\mathrm{L}u = 8e^t$ is

$$u_P(t) = 4t\,e^t\,.$$

Zero Degree Formula. For a forcing $f(t)$ with degree $d = 0$, characteristic $\mu + i\nu$, and multiplicity m that has the form

$$f(t) = \alpha e^{\mu t}\cos(\nu t) + \beta e^{\mu t}\sin(\nu t) = e^{\mu t}\,\mathrm{Re}\big((\alpha - i\beta)e^{i\nu t}\big)\,,$$

this formula gives the particular solution

$$u_P(t) = t^m e^{\mu t}\,\mathrm{Re}\left(\frac{\alpha - i\beta}{p^{(m)}(\mu + i\nu)}\,e^{i\nu t}\right)\,.$$

For this problem $f(t) = 8e^t$ and $p(z) = z^2 - 1$, so that $\mu + i\nu = 1$, $\alpha - i\beta = 8$, $m = 1$, and $p'(z) = 2z$, whereby

$$u_P(t) = t\,e^t\,\frac{8}{p'(1)} = t\,e^t\,\frac{8}{2} = 4t\,e^t\,.$$

Undetermined Coefficients. Because $m + d = m = 1$ and $\mu + i\nu = 1$, there is a particular solution in the form
$$u_P(t) = At\, e^t \,.$$
Because
$$u_P'(t) = A\big(t\, e^t + e^t\big)\,, \qquad u_P''(t) = A\big(t\, e^t + 2e^t\big)\,,$$
we see that
$$\mathrm{L}u_P(t) = u_P''(t) - u_P(t) = A\big(t\, e^t + 2e^t\big) - At\, e^t = 2Ae^t\,.$$
By setting $\mathrm{L}\big(u_P(t)\big) = 2Ae^t = 8e^t$, we see that $A = 4$. Therefore a particular solution of $\mathrm{L}u = 8e^t$ is
$$u_P(t) = 4t\, e^t\,.$$

(9) [10] The functions $1 + 2t$ and e^{2t} are solutions of the homogeneous equation
$$t\, x'' - (1 + 2t)x' + 2x = 0 \qquad \text{over } t > 0\,.$$
(You do not have to check that this is true!)
(a) [3] Show that these functions are linearly independent.
(b) [7] Give a general solution of the nonhomogeneous equation
$$t\, y'' - (1 + 2t)y' + 2y = \frac{8t^2}{1 + 2t} \qquad \text{over } t > 0\,.$$

Solution (a). The Wronskian of $1 + 2t$ and e^{2t} is
$$\mathrm{Wr}[1 + 2t, e^{2t}](t) = \det\begin{pmatrix} 1 + 2t & e^{2t} \\ 2 & 2e^{2t} \end{pmatrix} = (1 + 2t)2e^{2t} - 2e^{2t} = 4t\, e^{2t}\,.$$
Because $\mathrm{Wr}[1 + 2t, e^{2t}](t) \neq 0$ for $t > 0$, the functions $1 + 2t$ and e^{2t} are linearly independent.

Solution (b). Because the equation has *variable* coefficients, we must use either the general Green function method or the variation of parameters method to solve it. Because we are asked for a general solution, neither of these methods is favored. To apply either method we must first bring the equation into its normal form
$$y'' - \frac{1 + 2t}{t}\, y' + \frac{2}{t}\, y = \frac{8t}{1 + 2t} \qquad \text{over } t > 0\,.$$
Because $1 + 2t$ and e^{2t} are linearly independent, they constitute a fundamental set of solutions to the associated homogeneous equation.

Variation of Parameters. Because $1 + 2t$ and e^{2t} constitute a fundamental set of solutions to the associated homogeneous equation, we seek a general solution of the nonhomogeneous equation in the form
$$y(t) = (1 + 2t)u_1(t) + e^{2t}u_2(t)\,,$$
where $u_1'(t)$ and $u_2'(t)$ satisfy the linear algebraic system
$$(1 + 2t)u_1'(t) + e^{2t}u_2'(t) = 0\,,$$
$$2u_1'(t) + 2e^{2t}u_2'(t) = \frac{8t}{1 + 2t}\,.$$

The solution of this system is

$$u_1'(t) = -\frac{2}{1+2t}\,, \qquad u_2'(t) = 2e^{-2t}\,.$$

Integrate these equations to obtain

$$u_1(t) = c_1 - \log(1+2t)\,, \qquad u_2(t) = c_2 - e^{-2t}\,.$$

Therefore a general solution of the nonhomogeneous equation is

$$y(t) = (1+2t)c_1 + e^{2t}c_2 - (1+2t)\log(1+2t) - 1\,.$$

Remark. Another way to find $u_1'(t)$ and $u_2'(t)$ is to use the formulas

$$u_1'(t) = -\frac{Y_2(t)\,f(t)}{\mathrm{Wr}[Y_1,Y_2](t)}\,, \qquad u_2'(t) = \frac{Y_1(t)\,f(t)}{\mathrm{Wr}[Y_1,Y_2](t)}\,,$$

with $Y_1(t) = 1+2t$, $Y_2(t) = e^{2t}$, and $f(t) = 8t/(1+2t)$. They yield

$$u_1'(t) = -e^{2t}\,\frac{8t}{1+2t}\,\frac{1}{4t\,e^{2t}} = -\frac{2}{1+2t}\,,$$
$$u_2'(t) = (1+2t)\,\frac{8t}{1+2t}\,\frac{1}{4t\,e^{2t}} = 2e^{-2t}\,.$$

This approach reqires the memorization of two formulas. The General Green Function method requires the memorization of just one formula.

General Green Function. The Green function $G(t,s)$ is given by

$$G(t,s) = \frac{1}{\mathrm{Wr}[1+2s,e^{2s}](s)}\,\det\begin{pmatrix}1+2s & e^{2s}\\ 1+2t & e^{2t}\end{pmatrix} = \frac{e^{2t}(1+2s)-(1+2t)e^{2s}}{4s\,e^{2s}}\,.$$

The Green Function Formula then yields the particular solution

$$\begin{aligned} y_P(t) &= \int_0^t G(t,s)\,f(s)\,\mathrm{d}s = \int_0^t \frac{e^{2t}(1+2s)-(1+2t)e^{2s}}{4s\,e^{2s}}\,\frac{8s}{1+2s}\,\mathrm{d}s\\ &= 2e^{2t}\int_0^t e^{-2s}\,\mathrm{d}s - 2(1+2t)\int_0^t \frac{1}{1+2s}\,\mathrm{d}s\\ &= e^{2t}\big(1-e^{-2t}\big) - (1+2t)\log(1+2t)\,. \end{aligned}$$

Therefore a general solution of the nonhomogeneous equation is

$$y(t) = c_1(1+2t) + c_2e^{2t} + e^{2t} - 1 - (1+2t)\log(1+2t)\,.$$

Remark. Because the integrands are both continuous except at $s = -\frac{1}{2}$, and because we want our solution to be defined for every $t > 0$, the lower endpoint of integration in the Green Function Formula can be taken to be any $t_I > -\frac{1}{2}$. In that case both of the integrands are continuous over the interval of integration for every $t > 0$. We took $t_I = 0$ because it simplified the evaluation of the primitives at t_I. If we had been asked to solve an initial-value problem then we should have taken t_I to be the initial time.

Remark. Notice the general solutions produced by the Variation of Parameters and General Green Function methods differ slightly.

(10) [8] Give a real general solution of the equation

$$\mathrm{D}^2 v - 5\mathrm{D}v + 4v = 10\cos(3t)\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

Solution. This is a *nonhomogeneous* equation for $v(t)$ with *constant* coefficients. Its characteristic polynomial is

$$p(z) = z^2 - 5z + 4 = (z-1)(z-4)\,.$$

This has the simple real roots 1 and 4, which yields a real general solution of the associated homogeneous problem given by

$$v_H(t) = c_1 e^t + c_2 e^{4t}\,.$$

The forcing $10\cos(3t)$ has degree $d = 0$, characteristic $\mu + i\nu = i3$, and multiplicity $m = 0$. A particular solution $v_P(t)$ can be found by using either Key Identity Evaluations, the Zero Degree Formula, or Undetermined Coefficients. Below we show that each of these methods gives the particular solution

$$v_P(t) = -\tfrac{1}{5}\cos(3t) - \tfrac{3}{5}\sin(3t)\,.$$

Therefore a real general solution is

$$v = c_1 e^t + c_2 e^{4t} - \tfrac{1}{5}\cos(3t) - \tfrac{3}{5}\sin(3t)\,.$$

Key Indentity Evaluations. Because $m + d = m = 0$, we can simply evaluate the Key Identity at $z = \mu + i\nu = i3$, to find

$$\mathrm{L}\big(e^{i3t}\big) = p(i3)e^{i3t} = \big((i3)^2 - 5(i3) + 4\big)e^{i3t} = -(5 + i15)e^{i3t}\,.$$

Because the forcing is $10\cos(3t) = 10\,\mathrm{Re}\big(e^{i3t}\big)$, we divide the above by $1 + i3$ and multiply by -2 to find

$$\mathrm{L}\left(\frac{-2}{1+i3}\,e^{i3t}\right) = 10e^{i3t}\,.$$

Therefore a particular solution of $\mathrm{L}v = 10\cos(3t)$ is given by

$$\begin{aligned} v_P(t) = {} & \mathrm{Re}\left(\frac{-2}{1+i3}\,e^{i3t}\right) = -2\,\mathrm{Re}\left(\frac{1-i3}{1^2+3^2}\,e^{i3t}\right) = -\tfrac{1}{5}\,\mathrm{Re}\big((1-i3)e^{i3t}\big) \\ & = -\tfrac{1}{5}\,\mathrm{Re}\big((1-i3)\big(\cos(3t) + i\sin(3t)\big)\big) = -\tfrac{1}{5}\cos(3t) - \tfrac{3}{5}\sin(3t)\,. \end{aligned}$$

Zero Degree Formula. For a forcing $f(t)$ with degree $d = 0$, characteristic $\mu + i\nu$, and multiplicity m that has the form

$$f(t) = \alpha e^{\mu t}\cos(\nu t) + \beta e^{\mu t}\sin(\nu t) = e^{\mu t}\,\mathrm{Re}\big((\alpha - i\beta)e^{i\nu t}\big)\,,$$

this formula gives the particular solution

$$v_P(t) = t^m e^{\mu t}\,\mathrm{Re}\left(\frac{\alpha - i\beta}{p^{(m)}(\mu + i\nu)}\,e^{i\nu t}\right)\,.$$

For this problem $f(t) = 10\cos(3t)$ and $p(z) = z^2 - 5z + 4$, so that $\mu + i\nu = i3$, $\alpha - i\beta = 10$, $m = 0$, and $p(i3) = (i3)^2 - 5 \cdot (i3) + 4 = -9 + i15 + 4 = -5 - i15$. Therefore the particular solution of $\mathrm{L}v = 10\cos(3t)$ is given by

$$\begin{aligned} v_P(t) &= \mathrm{Re}\left(\frac{10}{-5 - i15}\, e^{i3t}\right) = \mathrm{Re}\left(\frac{-2}{1 + i3}\, e^{i3t}\right) = -2\,\mathrm{Re}\left(\frac{1 - i3}{1^2 + 3^2}\, e^{i3t}\right) \\ &= -\tfrac{1}{5}\,\mathrm{Re}\big((1 - i3)e^{i3t}\big) = -\tfrac{1}{5}\,\mathrm{Re}\big((1 - i3)\big(\cos(3t) + i\sin(3t)\big)\big) \\ &= -\tfrac{1}{5}\big(\cos(3t) + 3\sin(3t)\big) = -\tfrac{1}{5}\cos(3t) - \tfrac{3}{5}\sin(3t)\,. \end{aligned}$$

Undetermined Coefficients. Because $m + d = m = 0$ and $\mu + i\nu = i3$, there is a particular solution in the form

$$v_P(t) = A\cos(3t) + B\sin(3t)\,.$$

Because

$$v_P'(t) = -3A\sin(3t) + 3B\cos(3t)\,, \qquad v_P''(t) = -9A\cos(3t) - 9B\sin(3t)\,,$$

we see that

$$\begin{aligned} \mathrm{L}v_P(t) &= v_P''(t) - 5v_P'(t) + 4v_P(t) \\ &= \big(-9A\cos(3t) - 9B\sin(3t)\big) - 5\big(-3A\sin(3t) + 3B\cos(3t)\big) \\ &\quad + 4\big(A\cos(3t) + B\sin(3t)\big) \\ &= -(5A + 15B)\cos(3t) - (5B - 15A)\sin(3t)\,. \end{aligned}$$

After setting $\mathrm{L}v_P(t) = 10\cos(3t)$, the linear independence of $\cos(3t)$ and $\sin(3t)$ implies that

$$5A + 15B = -10\,, \qquad 5B - 15A = 0\,.$$

The solution of this linear algebraic system is $A = -\frac{1}{5}$ and $B = -\frac{3}{5}$. Therefore a particular solution of $\mathrm{L}v = 10\cos(3t)$ is given by

$$v_P(t) = -\tfrac{1}{5}\cos(3t) - \tfrac{3}{5}\sin(3t)\,.$$

(11) [10] The vertical displacement of a spring-mass system is governed by the equation

$$\ddot{h} + 10\dot{h} + 169h = a\cos(\omega t - \phi)\,,$$

where $a > 0$, $\omega > 0$, and $0 \leq \phi < 2\pi$.

(a) [2] Give the natural frequency and period of the system.
(b) [4] Show the system is under damped and give its damped frequency and period.
(c) [4] Find the steady state of the system and give its phasor.

Solution (a). If the units of time are seconds then the natural frequency is

$$\omega_o = \sqrt{169} = 13 \quad \text{rad/sec}\,.$$

The natural period is then

$$T_o = \frac{2\pi}{\omega_o} = \frac{2\pi}{13} \quad \text{sec}\,.$$

Solution (b). The characteristic polynomial of the equation is

$$p(z) = z^2 + 10z + 169 = (z + 5)^2 + 169 - 25 = (z + 5)^2 + 144 = (z + 5)^2 + 12^2\,.$$

This has the conjugate pair of roots $-5 \pm i12$. Therefore the system is *underdamped* and if the units of time are seconds then its damped frequency is

$$\omega_\eta = 12 \quad \text{rad/sec}\,.$$

The damped period is then

$$T_\eta = \frac{2\pi}{\omega_\eta} = \frac{2\pi}{12} \quad \text{sec}\,.$$

Solution (c). The forcing expressed in its phasor form is

$$a\cos(\omega t - \phi) = \mathrm{Re}\big(ae^{i(\omega t-\phi)}\big) = \mathrm{Re}\big(ae^{-i\phi}e^{i\omega t}\big)\,,$$

where its phasor is the complex number $ae^{-i\phi}$. The steady state of the system is its periodic solution. Because

$$p(i\omega) = (i\omega)^2 + 10(i\omega) + 169 = 169 - \omega^2 + i10\omega\,,$$

it is given by

$$h_P(t) = \mathrm{Re}\left(\frac{ae^{-i\phi}}{p(i\omega)}\,e^{i\omega t}\right) = \mathrm{Re}\left(\frac{ae^{-i\phi}}{169-\omega^2+i10\omega}\,e^{i\omega t}\right)\,.$$

Its phasor is the complex number

$$\frac{ae^{-i\phi}}{169-\omega^2+i10\omega}\,.$$

Remark. This solution is a simple harmonic oscillation with frequency ω. It is called the steady state solution of the nonhomogeneous equation because it is the only periodic solution of that equation. Every other solution of the nonhomogeneous equation approaches it as $t \to \infty$. This is because every solution of the associated homogeneous equation decays to zero as $t \to \infty$.

Remark. Below we show that this solution can also be found by Key Identity Evaluations, the Zero Degree Formula, or Undetermined Coefficients. Because the forcing can be expressed as

$$a\cos(\omega t - \phi) = a\cos(\phi)\cos(\omega t) + a\sin(\phi)\sin(\omega t)\,,$$

we see that it has degree $d = 0$, characteristic $\mu + i\nu = i\omega$, and multiplicity $m = 0$.

Key Identity Evaluations. Because $m + d = m = 0$, we can simply evaluate the Key Identity at $z = \mu + i\nu = i\omega$, to find

$$\mathrm{L}\big(e^{i\omega t}\big) = p(i\omega)e^{i\omega t} = \big((i\omega)^2 + 10(i\omega) + 169\big)e^{i\omega t} = \big(169 - \omega^2 + i10\omega\big)e^{i\omega t}\,.$$

We can multiply the Key Identity by $ae^{-i\phi}$ and divide by $169 - \omega^2 + i10\omega$ to obtain

$$\mathrm{L}\left(\frac{ae^{-i\phi}}{169-\omega^2+i10\omega}\,e^{i\omega t}\right) = ae^{-i\phi}e^{i\omega t}\,.$$

By real parts we see that a particular solution of $\mathrm{L}h = \mathrm{Re}\big(ae^{-i\phi}e^{i\omega t}\big)$ is

$$h_P(t) = \mathrm{Re}\left(\frac{ae^{-i\phi}}{169-\omega^2+i10\omega}\,e^{i\omega t}\right)\,.$$

Zero Degree Formula. For a forcing $f(t)$ with degree $d = 0$, characteristic $\mu + i\nu$, and multiplicity m that has the form

$$f(t) = \alpha e^{\mu t}\cos(\nu t) + \beta e^{\mu t}\sin(\nu t) = e^{\mu t}\,\mathrm{Re}\big((\alpha - i\beta)e^{i\nu t}\big)\,,$$

this formula gives the particular solution

$$h_P(t) = t^m e^{\mu t}\,\mathrm{Re}\left(\frac{\alpha - i\beta}{p^{(m)}(\mu + i\nu)}\,e^{i\nu t}\right)\,.$$

For this problem $f(t) = \mathrm{Re}\big(ae^{-i\phi}e^{i\omega t}\big)$ and $p(z) = z^2 + 10z + 169$, so that $\mu + i\nu = i\omega$, $\alpha - i\beta = ae^{-i\phi}$, $m = 0$, and

$$p(i\omega) = (i\omega)^2 + 10(i\omega) + 169 = 169 - \omega^2 + i10\omega\,.$$

Therefore the particular solution is

$$h_P(t) = \mathrm{Re}\left(\frac{ae^{-i\phi}}{p(i\omega)}\,e^{i\omega t}\right) = \mathrm{Re}\left(\frac{ae^{-i\phi}}{169 - \omega^2 + i10\omega}\,e^{i\omega t}\right)\,.$$

Undetermined Coefficients. Because $m + d = m = 0$ and $\mu + i\nu = i\omega$, there is a particular solution in the form

$$h_P(t) = A\cos(\omega t) + B\sin(\omega t) = \mathrm{Re}\big((A - iB)e^{i\omega t}\big)\,.$$

Because

$$h_P'(t) = -\omega A\sin(\omega t) + \omega B\cos(\omega t)\,, \qquad h_P''(t) = -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t)\,,$$

we see that

$$\begin{aligned}
\mathrm{L}h_P(t) &= h_P''(t) + 10h_P'(t) + 169h_P(t) \\
&= \big(-\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t)\big) + 10\big(-\omega A\sin(\omega t) + \omega B\cos(\omega t)\big) \\
&\quad + 169\big(A\cos(\omega t) + B\sin(\omega t)\big) \\
&= \big((169 - \omega^2)A + 10\omega B\big)\cos(\omega t) + \big(-10\omega A + (169 - \omega^2)B\big)\sin(\omega t)\,.
\end{aligned}$$

After setting $\mathrm{L}h_P(t) = \alpha\cos(\omega t) + \beta\sin(\omega t)$, where $\alpha = a\cos(\phi)$ and $\beta = a\sin(\phi)$, the linear independence of $\cos(\omega t)$ and $\sin(\omega t)$ implies that

$$(169 - \omega^2)A + 10\omega B = \alpha\,, \qquad -10\omega A + (169 - \omega^2)B = \beta\,.$$

The solution of this linear algebraic system is

$$A = \frac{(169 - \omega^2)\alpha - 10\omega\beta}{(169 - \omega^2)^2 + 100\omega^2}\,, \qquad B = \frac{10\omega\alpha + (169 - \omega^2)\beta}{(169 - \omega^2)^2 + 100\omega^2}\,.$$

Therefore the particular solution is given by

$$h_P(t) = \frac{(169 - \omega^2)\alpha - 10\omega\beta}{(169 - \omega^2)^2 + 100\omega^2}\,\cos(\omega t) + \frac{10\omega\alpha + (169 - \omega^2)\beta}{(169 - \omega^2)^2 + 100\omega^2}\,\sin(\omega t)\,.$$

Its phasor is $A - iB$, which is

$$\frac{(169 - \omega^2)\alpha - 10\omega\beta}{(169 - \omega^2)^2 + 100\omega^2} - i\,\frac{10\omega\alpha + (169 - \omega^2)\beta}{(169 - \omega^2)^2 + 100\omega^2}\,.$$

(12) [8] When a 10 gram mass is hung vertically from a spring, at rest it stretches the spring 20 cm. (Gravitational acceleration is $g = 980$ cm/sec^2.) A dashpot imparts a damping force of 280 dynes (1 dyne = 1 gram cm/sec^2) when the speed of the mass is 2 cm/sec. Assume that the spring force is proportional to displacement, that the damping force is proportional to velocity, and that there are no other forces. At $t = 0$ the mass is displaced 5 cm below its rest position and is released with a upward velocity of 4 cm/sec.

(a) [6] Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (DO NOT solve this initial-value problem, just write it down!)

(b) [2] Is this system undamped, under damped, critically damped, or over damped? (Give your reasoning!)

Solution (a). Let $h(t)$ be the displacement in centimeters at time t in seconds of the mass from its rest position, with upward displacements being positive. The governing initial-value problem then has the form

$$m\ddot{h} + c\dot{h} + kh = 0\,, \qquad h(0) = -5\,, \quad \dot{h}(0) = 4\,,$$

where m is the mass, c is the damping coefficient, and k is the spring constant. The problem says that $m = 10$ grams. The spring constant is obtained by balancing the weight of the mass $mg = 10 \cdot 980$ dynes) with the force applied by the spring when it is stetched 20 cm. This gives $k\,20 = 10 \cdot 980$, or

$$k = \frac{10 \cdot 980}{20} = 490 \quad \text{dynes/cm}\,.$$

The damping coefficient is obtained by balancing the force of 280 dynes with the damping force imparted by the dashpot when the speed of the mass is 2 cm/sec. This gives $c\,2 = 280$, or

$$c = \tfrac{280}{2} = 140 \quad \text{dynes sec/cm}\,.$$

Therefore the governing initial-value problem is

$$10\ddot{h} + 140\,\dot{h} + 490h = 0\,, \qquad h(0) = -5\,, \quad \dot{h}(0) = 4\,.$$

Remark. With the equation in normal form the answer is

$$\ddot{h} + 14\,\dot{h} + 49h = 0\,, \qquad h(0) = -5\,, \quad \dot{h}(0) = 4\,.$$

Remark. If we had chosen downward displacements to be positive then the governing initial-value problem would be the same except for the initial conditions, which would be $h(0) = 5$ and $\dot{h}(0) = -4$.

Solution (b). The governing differential equation has constant coefficients. Its characteristic polynomial is

$$p(z) = z^2 + 14z + 49 = (z + 7)^2\,,$$

which has the double real root -7. Therefore the system is *critically damped*.