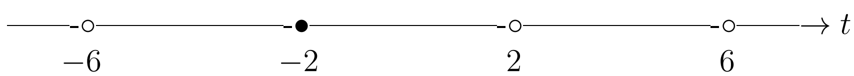
**Second In-Class Exam Solutions**
**Math 246, Professor David Levermore**
**Thursday, 15 March 2018**

(1) [4] Give the interval of definition for the solution of the initial-value problem

$$k''' - \frac{\cos(2t)}{6+t}\, k'' + \frac{e^{3t}}{2-t}\, k = \frac{6+t}{6-t}\,, \qquad k(-2) = k'(-2) = k''(-2) = 4\,.$$

**Solution.** The equation is linear and is already in normal form. Notice the following.
◇ The coefficient of $k''$ is undefined at $t = -6$ and is continuous elsewhere.
◇ The coefficient of $k$ is undefined at $t = 2$ and is continuous elsewhere.
◇ The forcing is undefined at $t = 6$ and is continuous elsewhere.
Plotting these points along with the inital time $t = -2$ on a time-line gives




Therefore the interval of definition is $(-6, 2)$ because:
- the initial time $t = -2$ is in $(-6, 2)$;
- all the coefficients and the forcing are continuous over $(-6, 2)$;
- the coefficient of $k''$ is undefined at $t = -6$;
- the coefficient of $k$ is undefined at $t = 2$.

**Remark.** All four reasons must be given for full credit.
∘ The first two reasons are why a (unique) solution exists over the interval $(-6, 2)$.
∘ The last two reasons are why this solution does not exist over a larger interval.

(2) [12] The functions $e^{3t}$ and $e^{-3t}$ are a fundamental set of solutions to $u'' - 9u = 0$.
(a) [8] Solve the general initial-value problem

$$u'' - 9u = 0\,, \qquad u(0) = u_0\,, \quad \dot{u}(0) = u_1\,.$$

(b) [4] Find the associated natural fundamental set of solutions to $u'' - 9u = 0$.

**Solution (a).** Because we are told that $e^{3t}$ and $e^{-3t}$ constitute a fundamental set of solutions to $u'' - 9u = 0$, we know that a general solution is

$$u(t) = c_1 e^{3t} + c_2 e^{-3t}\,.$$

Because

$$u'(t) = 3c_1 e^{3t} - 3c_2 e^{-3t}\,,$$

when the general initial conditions are imposed, we find that

$$u(0) = c_1 + c_2 = u_0\,, \qquad u'(0) = 3c_1 - 3c_2 = u_1\,.$$

Upon solving this linear algebraic systen we find that

$$c_1 = \frac{3u_0 + u_1}{6}\,, \qquad c_2 = \frac{3u_0 - u_1}{6}\,.$$

Therefore the solution of the general initial-value problem is

$$u(t) = \frac{3u_0 + u_1}{6}\, e^{3t} + \frac{3u_0 - u_1}{6}\, e^{-3t}\,.$$

**Solution (b).** The natural fundamental set of solutions associated with $t = 0$ is read off as the functions that multiply $u_0$ and $u_1$ in the solution of the general initial-value problem. Because that solution is

$$u(t) = \frac{3u_0 + u_1}{6}\, e^{3t} + \frac{3u_0 - u_1}{6}\, e^{-3t} = \frac{e^{3t} + e^{-3t}}{2}\, u_0 + \frac{e^{3t} - e^{-3t}}{6}\, u_1 \,,$$

we read off that the natural fundamental set of solutions associated with $t = 0$ is

$$N_0(t) = \frac{e^{3t} + e^{-3t}}{2} \,, \qquad N_1(t) = \frac{e^{3t} - e^{-3t}}{6} \,.$$

(3) [4] Suppose that $Z_1(t)$, $Z_2(t)$, and $Z_3(t)$ are solutions of the differential equation

$$z''' - 3z'' + (1 + t^2)z' + \sin(3t)z = 0 \,,$$

Suppose we know that $\mathrm{Wr}[Z_1, Z_2, Z_3](0) = 3$. Find $\mathrm{Wr}[Z_1, Z_2, Z_3](t)$.

**Solution.** The Abel Theorem says that $w(t) = \mathrm{Wr}[Z_1, Z_2, Z_3](t)$ satisfies $w' - 3w = 0$. It follows that $w(t) = ce^{3t}$ for some $c$. Because $w(0) = \mathrm{Wr}[Z_1, Z_2, Z_3](0) = 3$, this initial condition implies that $w(0) = ce^{3 \cdot 0} = 3$, whereby $c = 3$. Therefore $w(t) = 3e^{3t}$, which means that

$$\mathrm{Wr}[Z_1, Z_2, Z_3](t) = 3e^{3t} \,.$$

(4) [12] Let L be a linear ordinary differential operator with constant coefficients. Suppose that all the roots of its characteristic polynomial (listed with their multiplicities) are $-3 + i4$, $-3 + i4$, $-3 - i4$, $-3 - i4$, $-2$, $-2$, $-2$, $0$, $0$.

(a) [2] Give the order of L.
(b) [7] Give a real general solution of the homogeneous equation $\mathrm{L}v = 0$.
(c) [3] Give the degree, characteristic, and multiplicity for the forcing of the nonhomogeneous equation $\mathrm{L}w = t^2 e^{-2t}$.

**Solution (a).** Because there are 9 roots listed, the degree of the characteristic polynomial must be 9, whereby the order of L is 9.

**Solution (b).** A fundamental set of nine real-valued solutions is built as follows.

- ⋄ The conjugate pair of double roots $-3 \pm i4$ contributes

$$e^{-3t}\cos(4t) \,, \quad e^{-3t}\sin(4t) \,, \quad t\,e^{-3t}\cos(4t) \,, \quad \text{and} \quad t\,e^{-3t}\sin(4t) \,.$$

- ⋄ The triple real root $-2$ contributes

$$e^{-2t} \,, \qquad t\,e^{-2t} \,, \qquad \text{and} \qquad t^2 e^{-2t} \,.$$

- ⋄ The double real root 0 contributes 1 and $t$.

Therefore a real general solution of the homogeneous equation $\mathrm{L}v = 0$ is

$$\begin{aligned} v = {} & c_1 e^{-3t}\cos(4t) + c_2 e^{-3t}\sin(4t) + c_3 t\,e^{-3t}\cos(4t) + c_4 t\,e^{-3t}\sin(4t) \\ & + c_5 e^{-2t} + c_6 t\,e^{-2t} + c_7 t^2 e^{-2t} + c_8 + c_9 t \,. \end{aligned}$$

**Solution (c).** The forcing of the nonhomogeneous linear equation $\mathrm{L}w = t^2 e^{-2t}$ has degree $d = 2$ and characteristic $\mu + i\nu = -2$. Because the characteristic $\mu + i\nu = -2$ is listed as a triple root of the characteristic polynomial, it has multiplicity $m = 3$. Therefore, we have

$$d = 2 \,, \qquad \mu + i\nu = -2 \,, \qquad m = 3 \,.$$

(5) [8] What answer will be produced by the following Matlab commands?

```
>> ode = 'D2y – 6*Dy + 34*y = 5*exp(3*t)';
>> dsolve(ode, 't')

ans =
```

**Solution.** The commands ask Matlab for a real general solution of the equation

$$\mathrm{D}^2 y - 6\mathrm{D}y + 34y = 5e^{3t}\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

While your answer did not have to be given in Matlab format, Matlab will produce something equivalent to

```
exp(3*t)/5 + C1*exp(3*t)*cos(5*t) + C2*exp(3*t)*sin(5*t)
```

This can be seen as follows. This is a *nonhomogeneous* linear equation for $y(t)$ with *constant* coefficients. Its linear differential operator is $\mathrm{L} = \mathrm{D}^2 - 6\mathrm{D} + 34$. Its characteristic polynomial is

$$p(z) = z^2 - 6z + 34 = (z-3)^2 + 25 = (z-3)^2 + 5^2\,,$$

which has the conjugate pair of roots $3 \pm i5$. A real general solution of the associated homogeneous problem is

$$y_H(t) = c_1 e^{3t}\cos(5t) + c_2 e^{3t}\sin(5t)\,.$$

The forcing $5e^{3t}$ has degree $d = 0$, characteristic $\mu + i\nu = 3$, and multiplicity $m = 0$. A particular solution $y_P(t)$ can be found by using either Key Identity Evaluations, the Zero Degree Formula, or Undetermined Coefficients. Below we show that each of these methods gives the particular solution $y_P(t) = \frac{1}{5}e^{3t}$. Therefore a real general solution is

$$y = c_1 e^{3t}\cos(5t) + c_2 e^{3t}\sin(5t) + \tfrac{1}{5}e^{3t}\,.$$

Up to notational differences, this is the answer that Matlab produces.

**Key Identity Evaluations.** Because $m + d = m = 0$, we can simply evaluate the Key Identity at $z = \mu + i\nu = 3$, to find

$$\mathrm{L}(e^{3t}) = p(3)e^{3t} = (3^2 - 6\cdot 3 + 34)e^{3t} = 25e^{3t}\,.$$

Multiply this by $\frac{1}{5}$ to obtain $\mathrm{L}(\frac{1}{5}e^{3t}) = 5e^{3t}$. Hence, $y_P(t) = \frac{1}{5}e^{3t}$.

**Zero Degree Formula.** For a forcing $f(t)$ with degree $d = 0$, characteristic $\mu + i\nu$, and multiplicity $m$ that has the form

$$f(t) = \alpha e^{\mu t}\cos(\nu t) + \beta e^{\mu t}\sin(\nu t) = e^{\mu t}\,\mathrm{Re}\big((\alpha - i\beta)e^{i\nu t}\big)\,,$$

this formula gives the particular solution

$$y_P(t) = t^m e^{\mu t}\,\mathrm{Re}\left(\frac{\alpha - i\beta}{p^{(m)}(\mu + i\nu)}\,e^{i\nu t}\right)\,.$$

For this problem $f(t) = 5e^{3t}$ and $p(z) = z^2 - 6z + 34$, so that $\mu + i\nu = 3$, $\alpha - i\beta = 5$, and $m = 0$, whereby

$$y_P(t) = e^{3t}\,\frac{5}{p(3)} = \frac{5}{3^2 - 6\cdot 3 + 34}\,e^{3t} = \frac{5}{25}\,e^{3t} = \tfrac{1}{5}e^{3t}\,.$$

**Undetermined Coefficients.** Because $m + d = m = 0$ and $\mu + i\nu = 3$, there is a particular solution in the form

$$y_P(t) = Ae^{3t}\,.$$

Because

$$y_P'(t) = 3Ae^{3t}\,, \qquad y_P''(t) = 9Ae^{3t}\,,$$

we see that

$$\mathrm{L}y_P(t) = y_P''(t) - 6y_P'(t) + 34y_P(t) = [9Ae^{3t}] - 6[3Ae^{3t}] + 34[Ae^{3t}] = 25Ae^{3t}\,.$$

Setting $\mathrm{L}y_P(t) = 25Ae^{3t} = 5e^{3t}$, we see that $A = \frac{1}{5}$. Hence, $y_P(t) = \frac{1}{5}e^{3t}$.

(6) [8] Find a particular solution $w_P(t)$ of the equation $w'' - w = 8t\,e^t$.

**Solution.** This is a nonhomogeneous linear equation with constant coefficients. Its linear differential operator is $\mathrm{L} = \mathrm{D}^2 - 1$. Its characteristic polynomial is

$$p(z) = z^2 - 1 = (z+1)(z-1)\,,$$

which has two simple real roots $-1$ and $1$. The forcing $8t\,e^t$ has characteristic form with degree $d = 1$ and characteristic $\mu + i\nu = 1$, which has multiplicity $m = 1$. Therefore we can use either Key Identity Evaluations or Undetermined Coefficients to find a particular solution $w_P(t)$.

**Key Identity Evaluations.** Because $m = 1$ and $m + d = 2$ we need the first and second derivative with respect to $z$ of the Key Identity. The Key Identity and its first two derivatives with respect to $z$ are

$$\begin{aligned}
\mathrm{L}\big(e^{zt}\big) &= (z^2-1)e^{zt}\,,\\
\mathrm{L}\big(t\,e^{zt}\big) &= (z^2-1)t\,e^{zt} + 2ze^{zt}\,,\\
\mathrm{L}\big(t^2e^{zt}\big) &= (z^2-1)t^2e^{zt} + 4zt\,e^{zt} + 2e^{zt}\,.
\end{aligned}$$

By evaluating the first and second derivative of the Key Identity at $z = \mu + i\nu = 1$ we obtain

$$\mathrm{L}\big(t\,e^t\big) = 2e^t\,, \qquad \mathrm{L}\big(t^2e^t\big) = 4t\,e^t + 2e^t\,.$$

By subtracting the first equation from the second we find that

$$\mathrm{L}\big(t^2e^t - t\,e^t\big) = 4t\,e^t\,.$$

After multiplying this equation by 2 it becomes

$$\mathrm{L}\big(2t^2e^t - 2t\,e^t\big) = 8t\,e^t\,.$$

Therefore a particular solution of $\mathrm{L}w = 8t\,e^t$ is

$$w_P(t) = 2t^2e^t - 2t\,e^t = 2(t^2 - t)e^t\,.$$

**Undetermined Coefficients.** Because $m + d = 2$, $m = 1$ and $\mu + i\nu = 1$, there is a particular solution in the form

$$w_P(t) = (A_0t^2 + A_1t)\,e^t\,.$$

Because

$$
\begin{aligned}
w_P'(t) &= (A_0t^2 + A_1t)\,e^t + (2A_0t + A_1)\,e^t = \big(A_0t^2 + (2A_0 + A_1)t + A_1\big)\,e^t\,,\\
w_P''(t) &= \big(A_0t^2 + (2A_0 + A_1)t + A_1\big)\,e^t + \big(2A_0t + (2A_0 + A_1)\big)\,e^t\\
&= \big(A_0t^2 + (4A_0 + A_1)t + (2A_0 + 2A_1)\big)\,e^t
\end{aligned}
$$

we see that

$$
\begin{aligned}
\mathrm{L}w_P(t) &= w_P''(t) - w_P(t)\\
&= \big(A_0t^2 + (4A_0 + A_1)t + (2A_0 + 2A_1)\big)\,e^t - (A_0t^2 + A_1t)\,e^t\\
&= \big(4A_0t + 2(A_0 + A_1)\big)\,e^t\,.
\end{aligned}
$$

By setting $\mathrm{L}w_P(t) = 8t\,e^t$, the linear independence of $t\,e^t$ and $e^t$ implies that

$$
4A_0 = 8\,, \qquad A_0 + A_1 = 0\,,
$$

which yields $A_0 = 2$ and $A_1 = -2$. Therefore a particular solution of $\mathrm{L}w = 8t\,e^t$ is

$$
w_P(t) = (2t^2 - 2t)\,e^t\,.
$$

(7) [8] Compute the Green function $g(t)$ associated with the differential operator

$$
\mathrm{D}^2 + 4\mathrm{D} + 13\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.
$$

**Solution.** Because the linear differential operator has constant coefficients, its Green function $g(t)$ satisfies

$$
\mathrm{D}^2g + 4\mathrm{D}g + 13g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 1\,.
$$

The characteristic polynomial is

$$
p(z) = z^2 + 4z + 13 = (z + 2)^2 + 9 = (z + 2)^2 + 3^2\,,
$$

which has the conjugate pair of roots $-2 \pm i3$. Hence, the general solution of the equation is

$$
g(t) = c_1e^{-2t}\cos(3t) + c_2e^{-2t}\sin(3t)\,.
$$

The first initial condition implies $0 = g(0) = c_1$, whereby

$$
g(t) = c_2e^{-2t}\sin(3t)\,.
$$

Because

$$
g'(t) = 3c_2e^{-2t}\cos(3t) - 2c_2e^{-2t}\sin(3t)\,,
$$

the second initial condition implies $1 = g'(0) = 3c_2$, whereby $c_2 = \frac{1}{3}$. Therefore the Green function associated with the differential operator is

$$
g(t) = \tfrac{1}{3}e^{-2t}\sin(3t)\,.
$$

(8) [8] Solve the initial-value problem

$$x'' + 4x' + 13x = \frac{9e^{-2t}}{\sin(3t)}\,, \qquad x(\tfrac{\pi}{6}) = x'(\tfrac{\pi}{6}) = 0\,.$$

**Solution.** This is a nonhomogeneous linear equation with constant coefficients. Because its forcing does not have characteristic form, we cannot use either Key Identity Evaluations or Undetermined Coefficients. Because this is an initial-value problem with homogeneous initial conditions, we will use the Green function method.

By the previous problem the Green function for this problem is $g(t) = \frac{1}{3}e^{-2t}\sin(3t)$. Because the equation is in normal form, the initial time is $\frac{\pi}{6}$, and both of the initial values are 0, the solution to this inital-value problem is given by the Green formula

$$\begin{aligned} x(t) = \int_{\frac{\pi}{6}}^{t} g(t-s)f(s)\,\mathrm{d}s = \int_{\frac{\pi}{6}}^{t} e^{-2(t-s)}\sin\big(3(t-s)\big)\,\frac{9e^{-2s}}{\sin(3s)}\,\mathrm{d}s \\ = 3e^{-2t}\int_{\frac{\pi}{6}}^{t} \frac{\sin(3t-3s)}{\sin(3s)}\,\mathrm{d}s\,. \end{aligned}$$

By using the trig identity

$$\sin(3t-3s) = \sin(3t)\cos(3s) - \cos(3t)\sin(3s)\,,$$

we obtain

$$\begin{aligned} x(t) &= 3e^{-2t}\int_{\frac{\pi}{6}}^{t} \frac{\sin(3t)\cos(3s) - \cos(3t)\sin(3s)}{\sin(3s)}\,\mathrm{d}s \\ &= 3e^{-2t}\sin(3t)\int_{\frac{\pi}{6}}^{t} \frac{\cos(3s)}{\sin(3s)}\,\mathrm{d}s - 3e^{-2t}\cos(3t)\int_{\frac{\pi}{6}}^{t}\mathrm{d}s \\ &= e^{-2t}\sin(3t)\,\log\big(|\sin(3t)|\big) - 3e^{-2t}\cos(3t)\,\big(t - \tfrac{\pi}{6}\big)\,. \end{aligned}$$

**Remark.** The interval of definition for this initial-value problem is $(0, \frac{\pi}{3})$. Over this interval $\sin(3t)$ is positive. Therefore we could have written

$$x(t) = e^{-2t}\sin(3t)\,\log\big(\sin(3t)\big) - 3e^{-2t}\cos(3t)\,\big(t - \tfrac{\pi}{6}\big)\,.$$

**Remark.** This problem can also be solved by the general Green function method. However that approach is not as efficient because it does not use the fact the Green function $x(t)$ was already computed in the solution of the preceeding problem.

**Remark.** This problem can also be solved by using variation of parameters. However that approach is not as efficient because it does not directly solve the initial-value problem. Rather, after finding a particular solution the constants $c_1$ and $c_2$ in $x_H(t)$ must be determined to satisfy the initial conditions.

(9) [10] The functions $1 + 3t$ and $e^{3t}$ are solutions of the homogeneous equation

$$t\,p'' - (1+3t)p' + 3p = 0 \qquad \text{over } t > 0\,.$$

(You do not have to check that this is true!)

(a) [3] Show that these functions are linearly independent.

(b) [7] Give a general solution of the nonhomogeneous equation

$$t\,q'' - (1+3t)q' + 3q = \frac{27t^2}{1+3t} \qquad \text{over } t > 0\,.$$

**Solution (a).** The Wronskian of $1+3t$ and $e^{3t}$ is

$$\mathrm{Wr}[1+3t, e^{3t}](t) = \det\begin{pmatrix} 1+3t & e^{3t} \\ 3 & 3e^{3t} \end{pmatrix} = (1+3t)3e^{3t} - 3e^{3t} = 9t\,e^{3t}\,.$$

Because $\mathrm{Wr}[1+3t, e^{3t}](t) \neq 0$ for $t > 0$, the functions $1+3t$ and $e^{3t}$ are linearly independent.

**Solution (b).** Because the equation has *variable* coefficients, we must use either the general Green function method or the variation of parameters method to solve it. Because we are asked for a general solution, neither of these methods is favored. To apply either method we must first bring the equation into its normal form

$$q'' - \frac{1+3t}{t}\,q' + \frac{3}{t}\,q = \frac{27t}{1+3t} \qquad \text{over } t > 0\,.$$

Because $1+3t$ and $e^{3t}$ are linearly independent, they constitute a fundamental set of solutions to the associated homogeneous equation.

**Variation of Parameters.** Because $1+3t$ and $e^{3t}$ constitute a fundamental set of solutions to the associated homogeneous equation, we seek a general solution of the nonhomogeneous equation in the form

$$y(t) = (1+3t)u_1(t) + e^{3t}u_2(t)\,,$$

where $u_1'(t)$ and $u_2'(t)$ satisfy the linear algebraic system

$$\begin{aligned} (1+3t)u_1'(t) + e^{3t}u_2'(t) &= 0\,, \\ 3u_1'(t) + 3e^{3t}u_2'(t) &= \frac{27t}{1+3t}\,. \end{aligned}$$

The solution of this system is

$$u_1'(t) = -\frac{3}{1+3t}\,, \qquad u_2'(t) = 3e^{-3t}\,.$$

Integrate these equations to obtain

$$u_1(t) = c_1 - \log(1+3t)\,, \qquad u_2(t) = c_2 - e^{-3t}\,.$$

Therefore a general solution of the nonhomogeneous equation is

$$q(t) = (1+3t)c_1 + e^{3t}c_2 - (1+3t)\log(1+3t) - 1\,.$$

**Remark.** Another way to find $u_1'(t)$ and $u_2'(t)$ is to use the formulas

$$u_1'(t) = -\frac{Q_2(t)\,f(t)}{\mathrm{Wr}[Q_1, Q_2](t)}\,, \qquad u_2'(t) = \frac{Q_1(t)\,f(t)}{\mathrm{Wr}[Q_1, Q_2](t)}\,,$$

with $Q_1(t) = 1+3t$, $Q_2(t) = e^{3t}$, and $f(t) = 27t/(1+3t)$. They yield

$$u_1'(t) = -e^{3t}\,\frac{27t}{1+3t}\,\frac{1}{9t\,e^{3t}} = -\frac{3}{1+3t}\,, \qquad u_2'(t) = (1+3t)\,\frac{27t}{1+3t}\,\frac{1}{9t\,e^{3t}} = 3e^{-3t}\,.$$

This approach reqires the memorization of two formulas. The General Green Function method requires the memorization of just one formula.

**General Green Function.** The Green function $G(t,s)$ is given by

$$G(t,s) = \frac{1}{\mathrm{Wr}[1+3s, e^{3s}](s)} \det\begin{pmatrix} 1+3s & e^{3s} \\ 1+3t & e^{3t} \end{pmatrix} = \frac{e^{3t}(1+3s) - (1+3t)e^{3s}}{9s\, e^{3s}} \,.$$

The Green Formula then yields the particular solution

$$\begin{aligned} q_P(t) &= \int_0^t G(t,s)\, f(s)\, \mathrm{d}s = \int_0^t \frac{e^{3t}(1+3s) - (1+3t)e^{3s}}{9s\, e^{3s}} \, \frac{27s}{1+3s} \, \mathrm{d}s \\ &= 3e^{3t} \int_0^t e^{-3s}\, \mathrm{d}s - 3(1+3t) \int_0^t \frac{1}{1+3s} \, \mathrm{d}s \\ &= e^{3t}\big(1 - e^{-3t}\big) - (1+3t)\log(1+3t) \,. \end{aligned}$$

Therefore a general solution of the nonhomogeneous equation is

(1) $$q(t) = c_1(1+3t) + c_2 e^{3t} + e^{3t} - 1 - (1+3t)\log(1+3t) \,.$$

**Remark.** Because the integrands are both continuous except at $s = -\frac{1}{3}$, and because we want our solution to be defined for every $t > 0$, the lower endpoint of integration in the Green Formula can be any $t_I > -\frac{1}{3}$. We took $t_I = 0$ because it simplified the evaluation of the primitives at $t_I$. If we had been asked to solve an initial-value problem then we should have taken $t_I$ to be the initial time.

**Remark.** Notice the general solutions produced by the Variation of Parameters and General Green Function methods differ slightly. However the $c_1$ and $c_2$ that appear in each general solution are not the same.

(10) [8] Give a real general solution of the equation

$$\mathrm{D}^2 v - 5\mathrm{D}v + 4v = 10\cos(3t) \,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t} \,.$$

**Solution.** This is a *nonhomogeneous* equation for $v(t)$ with *constant* coefficients. Its linear differential operator is $\mathrm{L} = \mathrm{D}^2 - 5\mathrm{D} + 4$. Its characteristic polynomial is

$$p(z) = z^2 - 5z + 4 = (z-1)(z-4) \,.$$

This has the simple real roots 1 and 4, which yields a real general solution of the associated homogeneous problem given by

$$v_H(t) = c_1 e^t + c_2 e^{4t} \,.$$

The forcing $10\cos(3t)$ has degree $d = 0$, characteristic $\mu + i\nu = i3$, and multiplicity $m = 0$. A particular solution $v_P(t)$ can be found by using either Key Identity Evaluations, the Zero Degree Formula, or Undetermined Coefficients. Below we show that each of these methods gives the particular solution

$$v_P(t) = -\tfrac{1}{5}\cos(3t) - \tfrac{3}{5}\sin(3t) \,.$$

Therefore a real general solution is

$$v = c_1 e^t + c_2 e^{4t} - \tfrac{1}{5}\cos(3t) - \tfrac{3}{5}\sin(3t) \,.$$

**Key Identity Evaluations.** Because $m+d=m=0$, we can simply evaluate the Key Identity at $z=\mu+i\nu=i3$, to find

$$\mathrm{L}\big(e^{i3t}\big)=p(i3)e^{i3t}=\big((i3)^2-5(i3)+4\big)e^{i3t}=-(5+i15)e^{i3t}\,.$$

Because the forcing is $10\cos(3t)=10\,\mathrm{Re}\big(e^{i3t}\big)$, we divide the above by $1+i3$ and multiply by $-2$ to find

$$\mathrm{L}\left(\frac{-2}{1+i3}\,e^{i3t}\right)=10e^{i3t}\,.$$

Therefore a particular solution of $\mathrm{L}v=10\cos(3t)$ is given by

$$\begin{aligned}v_P(t)=&\ \mathrm{Re}\left(\frac{-2}{1+i3}\,e^{i3t}\right)=-2\,\mathrm{Re}\left(\frac{1-i3}{1^2+3^2}\,e^{i3t}\right)=-\tfrac{1}{5}\,\mathrm{Re}\big((1-i3)e^{i3t}\big)\\&=-\tfrac{1}{5}\,\mathrm{Re}\big((1-i3)\big(\cos(3t)+i\sin(3t)\big)\big)=-\tfrac{1}{5}\cos(3t)-\tfrac{3}{5}\sin(3t)\,.\end{aligned}$$

**Zero Degree Formula.** For a forcing $f(t)$ with degree $d=0$, characteristic $\mu+i\nu$, and multiplicity $m$ that has the form

$$f(t)=\alpha e^{\mu t}\cos(\nu t)+\beta e^{\mu t}\sin(\nu t)=e^{\mu t}\,\mathrm{Re}\big((\alpha-i\beta)e^{i\nu t}\big)\,,$$

this formula gives the particular solution

$$v_P(t)=t^m e^{\mu t}\,\mathrm{Re}\left(\frac{\alpha-i\beta}{p^{(m)}(\mu+i\nu)}\,e^{i\nu t}\right)\,.$$

For this problem $f(t)=10\cos(3t)$ and $p(z)=z^2-5z+4$, so that $\mu+i\nu=i3$, $\alpha-i\beta=10$, $m=0$, and $p(i3)=(i3)^2-5\cdot(i3)+4=-9+i15+4=-5-i15$. Therefore the particular solution of $\mathrm{L}v=10\cos(3t)$ is given by

$$\begin{aligned}v_P(t)=&\ \mathrm{Re}\left(\frac{10}{-5-i15}\,e^{i3t}\right)=\mathrm{Re}\left(\frac{-2}{1+i3}\,e^{i3t}\right)=-2\,\mathrm{Re}\left(\frac{1-i3}{1^2+3^2}\,e^{i3t}\right)\\&=-\tfrac{1}{5}\,\mathrm{Re}\big((1-i3)e^{i3t}\big)=-\tfrac{1}{5}\,\mathrm{Re}\big((1-i3)\big(\cos(3t)+i\sin(3t)\big)\big)\\&=-\tfrac{1}{5}\big(\cos(3t)+3\sin(3t)\big)=-\tfrac{1}{5}\cos(3t)-\tfrac{3}{5}\sin(3t)\,.\end{aligned}$$

**Undetermined Coefficients.** Because $m+d=m=0$ and $\mu+i\nu=i3$, there is a particular solution in the form

$$v_P(t)=A\cos(3t)+B\sin(3t)\,.$$

Because

$$v_P'(t)=-3A\sin(3t)+3B\cos(3t)\,,\qquad v_P''(t)=-9A\cos(3t)-9B\sin(3t)\,,$$

we see that

$$\begin{aligned}\mathrm{L}v_P(t)&=v_P''(t)-5v_P'(t)+4v_P(t)\\&=\big(-9A\cos(3t)-9B\sin(3t)\big)-5\big(-3A\sin(3t)+3B\cos(3t)\big)\\&\quad+4\big(A\cos(3t)+B\sin(3t)\big)\\&=-(5A+15B)\cos(3t)-(5B-15A)\sin(3t)\,.\end{aligned}$$

After setting $\mathrm{L}v_P(t)=10\cos(3t)$, the linear independence of $\cos(3t)$ and $\sin(3t)$ implies that

$$5A+15B=-10\,,\qquad 5B-15A=0\,.$$

The solution of this linear algebraic system is $A = -\frac{1}{5}$ and $B = -\frac{3}{5}$. Therefore a particular solution of $\mathrm{L}v = 10\cos(3t)$ is given by

$$v_P(t) = -\tfrac{1}{5}\cos(3t) - \tfrac{3}{5}\sin(3t)\,.$$

(11) [10] The vertical displacement of a spring-mass system is governed by the equation

$$\ddot{h} + 14\dot{h} + 625h = a\cos(\omega t - \phi)\,,$$

where $a > 0$, $\omega > 0$, and $0 \leq \phi < 2\pi$. Assume CGS units.

(a) [2] Give the natural frequency and period of the system.

(b) [4] Show the system is under damped and give its damped frequency and period.

(c) [4] Give the steady state solution in its phasor representation $\mathrm{Re}(\gamma\, e^{i\omega t})$.

**Solution (a).** The natural frequency is

$$\omega_o = \sqrt{625} = 25 \quad \text{rad/sec}\,.$$

The natural period is then

$$T_o = \frac{2\pi}{\omega_o} = \frac{2\pi}{\sqrt{625}} = \frac{2\pi}{25} \quad \text{sec}\,.$$

**Solution (b).** The characteristic polynomial of the equation is

$$\begin{aligned} p(z) = z^2 + 14z + 625 &= (z+7)^2 + 625 - 49 \\ &= (z+7)^2 + 576 = (z+7)^2 + 24^2\,. \end{aligned}$$

This has the conjugate pair of roots $-7 \pm i24$. Therefore the system is *under damped*. Its damped frequency is

$$\omega_\eta = \sqrt{576} = 24 \quad \text{rad/sec}\,.$$

The damped period is then

$$T_\eta = \frac{2\pi}{\omega_\eta} = \frac{2\pi}{\sqrt{576}} = \frac{2\pi}{24} = \frac{\pi}{12} \quad \text{sec}\,.$$

**Solution (c).** The forcing expressed in its phasor form is

$$a\cos(\omega t - \phi) = \mathrm{Re}\big(ae^{i(\omega t - \phi)}\big) = \mathrm{Re}\big(ae^{-i\phi}e^{i\omega t}\big)\,,$$

where its phasor is the complex number $ae^{-i\phi}$. It has degree 0 and characteristic $i\omega$, which has multiplicity 0. The steady state of the system is its periodic solution. Because

$$p(i\omega) = (i\omega)^2 + 14(i\omega) + 625 = 625 - \omega^2 + i14\omega\,,$$

a Key Identity evaluation shows that its phasor representation is

$$h_P(t) = \mathrm{Re}\left(\frac{ae^{-i\phi}}{p(i\omega)}\, e^{i\omega t}\right) = \mathrm{Re}\left(\frac{ae^{-i\phi}}{625 - \omega^2 + i14\omega}\, e^{i\omega t}\right)\,,$$

where its phasor $\gamma$ is the complex number

$$\gamma = \frac{ae^{-i\phi}}{625 - \omega^2 + i14\omega}\,.$$

(12) [8] When a 10 gram mass is hung vertically from a spring, at rest it stretches the spring 9.8 cm. (Gravitational acceleration is $g = 980$ cm/sec$^2$.) A dashpot imparts a damping force of 400 dynes (1 dyne = 1 gram cm/sec$^2$) when the speed of the mass is 2 cm/sec. Assume that the spring force is proportional to displacement, that the damping force is proportional to velocity, and that there are no other forces. At $t = 0$ the mass is displaced 4 cm above its rest position and is released with a downward velocity of 3 cm/sec.

(a) [6] Give an initial-value problem that governs the displacement $h(t)$ for $t > 0$. (DO NOT solve this initial-value problem, just write it down!)

(b) [2] Is this system undamped, under damped, critically damped, or over damped? (Give your reasoning!)

**Solution (a).** Let $h(t)$ be the displacement in centimeters at time $t$ in seconds of the mass from its rest position, with upward displacements being positive. Because there is no external forcing, the governing initial-value problem has the form

$$m\ddot{h} + c\dot{h} + kh = 0\,, \qquad h(0) = 4\,, \quad \dot{h}(0) = -3\,,$$

where $m$ is the mass, $c$ is the damping coefficient, and $k$ is the spring constant. The problem says that $m = 10$ grams. The damping coefficient $c$ is found by equating the damping force imparted by the dashpot when the speed of the mass is 2 cm/sec, which is $c\,2$ dynes, with the force of 400 dynes. This gives $c\,2 = 400$, or

$$c = \frac{400}{2} = 200 \quad \text{dynes sec/cm}\,.$$

The spring constant $k$ is found by equating the force of the spring when it is stetched 9.8 cm, which is $k\,9.8$ dynes, with the weight of the mass, which is $mg = 10 \cdot 980$ dynes. This gives $k\,9.8 = 10 \cdot 980$, or

$$k = \frac{10 \cdot 980}{9.8} = 1000 \quad \text{dynes/cm}\,.$$

Therefore the governing initial-value problem is

$$10\ddot{h} + 200\,\dot{h} + 1000h = 0\,, \qquad h(0) = 4\,, \quad \dot{h}(0) = -3\,.$$

**Remark.** With the equation in normal form the answer is

$$\ddot{h} + 20\,\dot{h} + 100h = 0\,, \qquad h(0) = 4\,, \quad \dot{h}(0) = -3\,.$$

**Remark.** If we had chosen downward displacements to be positive then the governing initial-value problem would be the same except for the initial conditions, which would be $h(0) = -4$ and $\dot{h}(0) = 3$.

**Solution (b).** The governing differential equation has constant coefficients. The normal form of its linear differential operator is $\mathrm{D}^2 + 20\mathrm{D} + 100$. Its characteristic polynomial is

$$p(z) = z^2 + 20z + 100 = (z + 10)^2\,,$$

which has the double real root $-10$. Therefore the system is *critically damped.*

**Alternative Solution (b).** Because the normal form of the differential equation is $\ddot{h} + 20\,\dot{h} + 100h = 0$, we see that the damping rate is $\eta = 20/2 = 10$ and the natural frequency is $\omega_0 = \sqrt{100} = 10$. Because $\eta = 10 = \omega_0$, the system is *critically damped.*