Math 246 Fall 2017 Final Exam

- Be sure your name, section number and problem number are on each answer sheet and that you have copied and signed the honor pledge on the first answer sheet.
- Follow the instructions as to which problem goes on which answer sheet. You may use the backs of the answer sheets. You may ask for an additonal answer sheet if needed.
- No calculators, cell phone or formula sheets are permitted. A Laplace Transform table can be found at the bottom of the back side of this page.
- For problems with multiple parts, whether the parts are related or not, be sure to go on to subsequent parts whenever possible even if there is some part you cannot do.
- Integrals (if any) do need to be evaluated but numerical answers do not need to be simplified.

Please put problem 1 on answer sheet 1

1. (a) [14] Solve the following initial-value problem and determine the interval of definition of the solution

$$(t^2+1)y' + 4ty = 4t\,, \qquad y(0) = 3\,.$$

(b) [14] Determine constants a and b such that the following equation is exact. Then find the general solution in implicit form:

$$(axy - y^3)\,\mathrm{d}x + (4y + 3x^2 - bxy^2)\,\mathrm{d}y = 0\,.$$

Please put problem 2 on answer sheet 2

2. (a) Consider the autonomous differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = (y-1)^2(y^2-2y)\,.$$

i. [6] Determine all stationary (equilibrium) solutions and sketch the phase-line portrait.

ii. [4] Let $y_1(t)$ be the solution of the above ODE with initial condition $y_1(0) = 0.5$ and let $y_2(t)$ be another solution satisfying $y_2(0) = 1.5$. Find the following limit:

$$\lim_{t\to\infty}\big(y_2(t) - y_1(t)\big)\,.$$

(b) Consider the initial value problem

$$y' = 2ty(y-1)\,, \qquad y(0) = -1\,.$$

i. [8] Use the explicit Euler method with two steps to compute an approximation of $y(2)$.

ii. [12] Find the explicit solution of the initial-value problem.

Please put problem 3 on answer sheet 3

3. (a) [14] Solve the initial-value problem $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}^{\mathrm{I}}$ where

$$\mathbf{A} = \begin{pmatrix} -1 & -2 \\ 4 & -5 \end{pmatrix}, \qquad \mathbf{x}^{\mathrm{I}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(b) Two interconnected tanks, each with a capacity of 75 liters, contain brine (salt water). Initially the first tank contains 25 liters and the second tank contains 40 liters. Brine with a salt concentration of 7 grams per liter flows into the first tank at 5 liters per hour. Well-stirred brine flows from the first tank into the second at 6 liters per hour, from the second into the first at 4 liters per hour, from the first into a drain at 2 liter per hour, and from the second into a drain at 3 liters per hour. Initially there are 19 grams of salt in the first tank and 32 grams in the second tank.

i. [4] Determine the volume of brine in each tank at time $t > 0$.

ii. [8] Give an initial-value problem that governs the grams of salt in each tank as a function of time. **(Do not solve the IVP.)**

iii. [2] Give the interval of definition for the solution of this initial-value problem.

Exam Continues on Other Side

Please put problem 4 on answer sheet 4

4. (a) [18] Find the general solution of the differential equation

$$2y'' + 3y' + y = t + 2\sin(t)\,.$$

(b) [12] Given that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are solutions of the associated homogeneous differential equation, find the general solution of

$$t^2 y'' - 2y = \frac{1}{t}\,, \quad t > 0\,.$$

Please put problem 5 on answer sheet 5

5. (a) [14] Find the Laplace transform of the solution of the following initial-value problem. **(Do not solve the IVP.)**

$$y'' - 2y' + y = te^t + 4\,, \qquad y(0) = 1\,, \quad y'(0) = 1\,.$$

(b) [14] Find the inverse Laplace transform of $Y(s) = \dfrac{e^{-2s}}{s^2 + s - 2}$.

Please put problem 6 on answer sheet 6

6. A real 2×2 matrix $\mathbf{B}$ has the eigenpairs

$$\left(-1\,, \begin{pmatrix} 3 \\ -2 \end{pmatrix}\right) \quad \text{and} \quad \left(-2\,, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right)\,.$$

(a) [5] Give a general solution to the linear planar system $\mathbf{x}' = \mathbf{B}\mathbf{x}$.

(b) [5] Give an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ that diagonalize $\mathbf{B}$.

(c) [10] Compute $e^{t\mathbf{B}}$.

(d) [8] Sketch a phase-plane portrait for this system and identify its type. Classify the stability of the origin. Carefully mark all sketched orbits with arrows!

Please put problem 7 on answer sheet 7

7. Consider the nonlinear planar system

$$\dot{p} = -3p + q\,, \qquad \dot{q} = -2p - q - 5p^2\,.$$

(a) [6] This system has two stationary points. Find them.

(b) [6] Find the Jacobian matrix at each stationary point.

(c) [8] Classify the type and stability of each stationary point.

(d) [8] Sketch the phase-plane portrait of the system near each stationary point. Carefully mark all sketched orbits with arrows!

Table of Laplace Transforms

$\mathcal{L}[c] = \frac{c}{s}$	$\mathcal{L}[e^{at}\sin(bt)] = \frac{b}{(s-a)^2+b^2}$
$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$	$\mathcal{L}[y'(t)] = s\mathcal{L}[y(t)] - y(0) = sY(s) - y(0)$
$\mathcal{L}[e^{at}] = \frac{1}{s-a}$	$\mathcal{L}[y''(t)] = s^2\mathcal{L}[y(t)] - sy(0) - y'(0) = s^2Y(s) - sy(0) - y'(0)$
$\mathcal{L}[\cos(bt)] = \frac{s}{s^2+b^2}$	$\mathcal{L}[u(t-c)] = \mathcal{L}[u_c(t)] = \frac{e^{-cs}}{s}$
$\mathcal{L}[\sin(bt)] = \frac{b}{s^2+b^2}$	$\mathcal{L}[u(t-c)j(t-c)] = \mathcal{L}[u_c(t)j(t-c)] = e^{-cs}\mathcal{L}[j(t)]$
$\mathcal{L}[e^{at}t^n] = \frac{n!}{(s-a)^{n+1}}$	$\mathcal{L}[\sinh(at)] = \frac{a}{s^2-a^2}$
$\mathcal{L}[e^{at}\cos(bt)] = \frac{s-a}{(s-a)^2+b^2}$	$\mathcal{L}[\cosh(at)] = \frac{s}{s^2-a^2}$