Math 246, Jeffrey Adams

Test I, Monday, September 24, 2018 SOLUTIONS

No calculators, notes, etc. Do one problem per page, and be sure to put your name and problem number at the top of each page. For full credit show your work. Each problem is worth 20 points.

Problem 1.
a. Solve the initial value problem:

$$\frac{dy}{dt} = \sin(t)y^2, \quad y(0) = 2$$

b. Determine the interval of definition for the solution if the initial–value problem

$$\frac{dy}{dt} + \frac{1}{t^2-1}y = \frac{1}{t^2-9}, \quad y(2) = 0$$

(It is *not* necessary to solve the equation.)

Solution:
(a) The equation is separable.

$$\begin{aligned}
\frac{dy}{y^2} &= \sin(t)\,dt \\
\int \frac{dy}{y^2} &= \int \sin(t)\,dt \\
\int y^{-2}\,dy &= \int \sin(t)\,dt \\
-y^{-1} &= -\cos(t) + C \\
y^{-1} &= \cos(t) + C \\
y &= \frac{1}{\cos(t) + C}
\end{aligned}$$

Plugging in $t = 0, y = 2$ gives $2 = \frac{1}{1+C}$, so $1 + C = \frac{1}{2}$, and $C = -\frac{1}{2}$. The solution is

$$\boxed{y = \frac{1}{\cos(t) - \frac{1}{2}}}$$

(b) The equation is linear and in normal form, so the only issue is where the functions of t are not defined. These points are $t = \pm 1, \pm 3$. The initial value problem has $t_0 = 2$, and since $1 < 2 < 3$, the solution is valid on

$$\boxed{1 < t < 3}$$

Problem 2. Consider the differential equation

$$\frac{dy}{dt} = \frac{(y-1)^2(y-3)}{(y+2)}$$

(a) What are the stationary solutions?

(b) Sketch the phase-line portrait. Identify each stationary solution as being stable, semi-stable, or unstable.

(c) If $y(0) = 2$, what is $\lim_{t\to\infty} y(t)$?

Solution:
(a) The stationary solutions are where the function is 0, i.e. $y = 1, 3$. Note that $y = 2$ is a singularity but not a stationary solution.
(b) The function changes sign at $t = 3$ and $t = -2$, but not $t = 1$. The phase line portrait is therefore

$$+++++[-2]-----[1]-----[3]+++++$$

Therefore $y = 1$ is semistable, and $y = 3$ is unstable.
(c) Since $1 < 2 < 3$, and the sign is $-$ $\lim_{t\to\infty} y(t) = 1$.

Problem 3. Consider the initial value problem

$$\frac{dy}{dt} = y - t^3, y(0) = 1$$

Use the (explicit) Euler method with $h = 1$ to estimate $y(2)$.

Solution: Recall the formula, for step size h, and $f(t, y) = y - t^3$, is

$$Y(t_{i+1}) \simeq Y(t_i) + hf(t_i, Y(t_i))$$

We start with $t_0 = 0$, and then $t_1 = 1, t_2 = 2$. The initial condition is $Y(0) = 1$, so

$$\begin{aligned} Y(1) &\simeq Y(0) + hf(0, Y(0)) \\ &= 1 + 1f(0, 1) \\ &= 1 + 1(1 - 0) = 2 \end{aligned}$$

and

$$\begin{aligned} Y(2) &\simeq Y(1) + hf(1, Y(1)) \\ &= 2 + 1f(1, 2) \\ &= 2 + 1(2 - 1) \\ &= 3 \end{aligned}$$

Problem 4. Consider the differential equation

$$(y + 3x^3)dx + (x\ln(x) - 4x)dy = 0$$

in the region $x > 0$.

(a) Show that the equation is not exact.

(b) Show that $\mu(x) = \frac{1}{x}$ is an integrating factor.

(c) Give the general solution of the equation. (For full credit give an explicit solution.)

Solution:

(a) $M_y = 1$ and $N_x = \ln(x) + 1 - 4 = \ln(x) - 3$. These aren't equal.

(b) Replace the equation with

$$\frac{1}{x}(y + 3x^3)dx + \frac{1}{x}(x\ln(x) - 4x)dy = 0$$

which simplifies to

$$(\frac{y}{x} + 3x^2)dx + (\ln(x) - 4)dy = 0$$

and now (with the new M, N) we have

$$M_y = \frac{1}{x}, N_x = \frac{1}{x}$$

and these are equal.

(c) Let $H = H(x, y)$ and solve

$$H_x = \frac{y}{x} + 3x^2$$

which gives

$$H = y\ln(x) + x^3 + C(y).$$

Now take $\frac{d}{dy}$ of both sides and set this equal to N:

$$\ln(x) + c'(y) = \ln(x) - 4$$

so $C(y) = -4y$ and the implicit solution is

$$y \ln(x) + x^3 - 4y = C$$

This is easy to solve for y:

$$\boxed{y = \frac{C - x^3}{\ln(x) - 4}}$$

Problem 5.
In the absence of predators a certain population of ants doubles every 7 weeks. There are currently 100,000 ants. There are also anteaters, that eat 1,000 ants per week.
(a) Write the differential equation describing the population of ants as a function of time.
(b) Solve the equation in (a). (You should not have any undetermined constants in your answer.)

Solution:
(a) Without predators the population $P(t)$ satisfies $\frac{dP}{dt} = rP$, so $P(t) = Ce^{rt}$. Note that $P(0) = C$, and $P(7) = Ce^{7r}$. Since the doubling time is 7 weeks this gives $P(7) = 2P(0)$, and plugging in the values gives $Ce^{7r} = 2C$, or $e^{7r} = 2$, so $7r = \ln(2)$, and $r = \frac{\ln(2)}{7}$. We also know $C = P(0) = 100{,}000$.

Now with the predators the differential equation is

$$\frac{dP}{dt} = rP - 1{,}000$$

and $r = \frac{\ln(2)}{7}$, so

$$\boxed{\frac{dP}{dt} = \frac{\ln(2)}{7}P - 1{,}000}$$

(b) Write the equation as $\frac{dP}{dt} - \frac{\ln(2)}{7}P = -1{,}000$. To save ink write this as

$$\frac{dP}{dt} - rP = k$$

where $r = \frac{\ln(2)}{7}$ and $k = -1{,}000$.

The integrating factor is $e^{\int -rdt} = e^{-rt}$, so the equation becomes

$$e^{-rt}(\frac{dP}{dt} - rP) = ke^{-rt}$$

The left hand side is

$$\frac{d}{dt}Pe^{-rt} = ke^{-rt}$$

and integrating both sides gives

$$Pe^{-rt} = -\frac{k}{r}e^{-rt} + D$$

(not the same constant as in (a)) or

$$P(t) = De^{rt} - \frac{k}{r}$$

Now plug k, r back in:

$$P(t) = De^{\frac{\ln(2)}{7}t} + \frac{7{,}000}{\ln(2)}$$

Setting $P(0) = 100{,}000$, gives

$$100,000 = D + \frac{7{,}000}{\ln(2)}$$

so $D = 100{,}000 - \frac{7,}{\ln(2)}$ and the solution is

$$\boxed{P(t) = (100{,}000 - \frac{7,}{\ln(2)})e^{\frac{\ln(2)}{7}} + \frac{7{,}000}{\ln(2)}}$$