Math 246, Jeffrey Adams

Test 2, Friday, October 19, 2018
SOLUTIONS

No calculators, notes, etc. Do one problem per page, and be sure to put your name and problem number at the top of each page. For full credit show your work.

Problem 1 [20 points]

Suppose L is an ordinary differential operator with constant coefficients. Suppose the characteristic polynomial of L is

$$p(x) = (x^2 - 6x + 10)^2(x^2 - 1)x^3.$$

What is the general (real-valued) solution to the differential equation $Ly = 0$?

Solution:

The roots of $x^2 - 6x + 10$ are $\frac{1}{2}(6 \pm \sqrt{36 - 40}) = \frac{1}{2}(6 \pm \sqrt{-4}) = 3 \pm i$, and $x^2 - 1 = (x + 1)(x - 1)$. So the terms are:

$(x^2 - 6x + 10)^2$:

Since the term is squared you bring down a power of t:

$$e^{3t}\cos(t), e^{3t}\sin(t), te^{3t}\cos(t), te^{3t}\sin(t),$$

$x^2 - 1 = (x + 1)(x - 1)$: e^t, e^{-t}

x^3: $1, t, t^2$

So the general solution is

$$c_1e^{3t}\cos(t)+tc_2e^{3t}\cos(t)+c_3e^{3t}\sin(t)+tc_4e^{3t}\sin(t)+c_5e^t+c_6e^{-t}+c_7t^2+c_8t+c_9$$

Problem 2 [25 points]

Consider the differential equation

$$(*) \qquad y'' - 2y' + y = t$$

(a) Find the general solution to the associated homogeneous equation

$$y'' - 2y' + y = 0.$$

(b) Find the general solution (*).

(c) Find the solution to (*) satisfying $y(0) = 1, y'(0) = 2$.

Solution:
(a) The characteristic equation is $x^2 - 2x + 1 = (x-1)^2$, so the general solution is $c_1 e^t + c_2 t e^t$
(b) We need a particular solution to (*). We have $d = 1, \mu + i\nu = 0, m = 0$. Key identity: $L(e^{zt}) = p(z)e^{zt}$, set $z = 0$: $L(e^0) = p(0) = 1$, i.e. $L(1) = 1$.
$L(te^{zt}) = (p(z) + p'(z)t)e^{zt}$, set $z = 0$: $L(t) = (p'(0) + p(0)t) = -2 + t$.
These are all the terms we need. We see $L(A*1+B*t) = A+B(-2+t)$. Set this equal to t. This gives $B = 1$ and $A - 2B = 0$, i.e. $A = 2$. So the particular solution is $2*1+1*t = 2+t$.

$$y_p = 2 + t$$

Undetermined coefficients: since $d = 1, m = 0, \mu + i\nu = 0$ the solution is of the form $a + bt$. Then $L(a + bt) = L(a) + bL(t) = a + b(-2 + t)$, so set this equal to t, giving $b = 1$ and then $a - 2b = 0$ gives $a = 2$, so $y_p = 2 + t$ again.
The general solution is

$$y = c_1 e^t + c_2 t e^t + 2 + t$$

(c) $y(0) = c_1 + 2$; $y'(t) = c_1 e^t + c_2 e^t + c2te^t + 1$, so $y'(0) = c_1 + c_2 + 1$. So the initial conditions give

$$c_1 + 2 = 1$$
$$c_1 + c_2 + 1 = 2$$

which gives $c_1 = -1, c_2 = 2$ so the solution to the initial value problem is

$$y = -e^t + 2te^t + 2 + t$$

Problem 3 [20 points]
(a) Suppose you want to use the Green function method to solve

$$y'' + y = e^{\sin(t)}.$$

What is the Green function $g(t)$?
(b) Suppose $Ly = 0$ is a linear homogeneous differential equation, and $Y_1(t), \ldots, Y_n(t)$ are solutions. Suppose the Wronskian $W(Y_1, \ldots, Y_n)(0) = 0$. What is $W(Y_1, \ldots, Y_n)(1)$? Justify your answer.

Solution:
(a) $g(t)$ is the solution to $y'' + y = 0, y(0) = 0, y'(0) = 1$, which is $\cos(t)$.
(b) The Wronskian of the solution to a differential equation is either identically 0, or never 0. Since $W(Y_1, \ldots, Y_n)(0) = 0$, the Wronskian is identically 0, so $W(Y_1, \ldots, Y_n)(1) = 0$.
Problem 4 [20 points]

The vertical displacement of a mass on a spring is given by

$$h'' + ch' + 9h = 0$$

for some (positive) damping constant c.
(a) For what values of c is the system under damped?
(b) For what values of c is the system over damped?
(c) For what value of c is the system critically damped?
(d) Give a *rough* sketch of the solution to the critically damped equation satisfying with $y(0) = y'(0) = 1$.

Solution:
(a) The characteristic equation is $x^2 + cx + 9$, which has roots $\frac{1}{2}(-c \pm \sqrt{c^2 - 36})$. This is under damped if there are two complex roots, i.e. $c^2 - 36 < 0$, i.e. $c < 6$.
(b) This is over damped if there are two real roots, i.e. $c > 6$.
(c) This is critically damped if $c = 6$.

Problem 5 [15 points]

Consider the matlab commands:

```
>> syms y(t)
>> ode='D2y+3*Dy+y=e^cos(t)';
>> dsolve(ode,'y(0)=1', 'Dy(0)=2')
```

What is the initial value problem this code is solving?

Solution:

$$y'' + 3y' + y = e^{\cos(t)}, y(0) = 1, y'(0) = 2$$