Math 246, Jeffrey Adams

Test 3, Friday, November 16, 2018 SOLUTIONS

Problem 1
(a) Convert the linear differential equation

$$ty''' + e^t y'' - \frac{1}{t}y' + 6y = \sin(t)$$

into a single first order matrix equation.
(b) Find the inverse Laplace transform $\mathcal{L}^{-1}[X(s)](t)$ of the function

$$X(s) = \frac{3s + 11}{s^2 - 3s - 4}$$

Solution:
(a)

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{6}{t} & \frac{1}{t^2} & -\frac{e^t}{t} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \sin(t) \end{pmatrix}$$

(b)
Solution
$s^2 - 3s - 4 = (s+1)(s-4)$, so

$$\frac{3s+11}{s^2 - 3s - 4} = \frac{A}{s+1} + \frac{B}{s-4}$$

for some A, B. Combining over a common denominator gives

$$\frac{3s+11}{s^2 - 3s - 4} = \frac{A(s-4) + B(s+1)}{s^2 - 3s - 4}$$

or

$$\frac{3s+11}{s^2 - 3s - 4} = \frac{(A+B)s + (-4A+B)}{s^2 - 3s - 4}$$

So $A + B = 2$ and $4a + B = 11$, which gives

$$A = \frac{-8}{5}, \quad B = \frac{23}{5}$$

So the inverse Laplace transform is

$$\mathcal{L}^{-1}(\frac{-\frac{8}{5}}{s+1}) + \mathcal{L}^{-1}(\frac{\frac{23}{5}}{s-4})$$

By the table we conclude this equals

$$-\frac{8}{5}e^{-t} + \frac{23}{5}e^{4t}$$

Problem 2 Find the general solution of

$$ty'' + y' = 1$$

You can use the fact that the solutions to $ty'' + y' = 0$ are 1 and $\ln(t)$.
Solution:

The normal form is $y'' + \frac{1}{t}y' = \frac{1}{t}$. Variation of parameters gives $y = u_1(t) + u_2(t)\ln(t)$, where

$$u_1' + u_2' \ln(t) = 0$$
$$u_1' * 0 + u_2' \frac{1}{t} = \frac{1}{t}.$$

This gives $u_2' = 1$, so $u_2 = t$. Then $u_1' = -\ln(t)$, so $u_1 = -(t\ln(t) - t)$. Then

$$y = -(t\ln(t) - t) * 1 + t(\ln(t)) = t$$

is a particular solution, and the general solution is

$$c_1 + c_2 \ln(t) + t$$

Problem 3

Find the general real solution of the system

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix} \vec{x}(t)$$

Solution:

The characteristic polynomial is $(-3-z)(-1-z)+1 = z^2+3z+z+3+1 = z^2 + 4z + 4 = (z+2)^2$. The -2 eigenspace is given by

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So $\vec{v} = (1,1)$. So one solution is $\vec{x}_1(t) = e^{-2t}(1,1)$. Take $\vec{w} = (1,0)$ to give another solution:

$$\begin{aligned} e^{-2t}(\vec{w} + (A+2I)t\vec{w}) &= e^{-2t}(\begin{pmatrix}1\\0\end{pmatrix} + t\begin{pmatrix}-1 & 1\\-1 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}) \\ &= e^{-2t}(\begin{pmatrix}1\\0\end{pmatrix} + t\begin{pmatrix}-1\\-1\end{pmatrix}) \\ &= e^{-2t}\begin{pmatrix}1-t\\-t\end{pmatrix} \end{aligned}$$

So the general solution is

$$c_1e^{-2t}\begin{pmatrix}1\\0\end{pmatrix} + c_2e^{-2t}\begin{pmatrix}1-t\\-t\end{pmatrix}$$

There are other ways to solve this problem.

Problem 4

Consider the initial value problem

$$y'' + 4y' - 3y = f(t) \quad y(0) = 2,\, y'(0) = 1$$

where

$$f(t) = \begin{cases} t & 0 \le t \le 1 \\ e^t & 1 < t \end{cases}$$

(a) Compute the Laplace transform of $f(t)$

(b) Determine the Laplace transform $Y(s)$ of the solution $y(t)$ of the equation. It is *not* necessary to find the inverse Laplace transform $y(t)$.

Solution:

(a) The function is

$$f(t) = t(u(t) - u(t-1) + e^t u(t-1) = tu(t) + (-t + e^t)u(t-1)$$

So

$$\mathcal{L}(f)(s) = \mathcal{L}(t) + \mathcal{L}((e^t - t)u(t-1)$$

and $\mathcal{L}(t) = \frac{1}{s^2}$. Also

$$\mathcal{L}((e^t - 1)u(t-1)) = \mathcal{L}(u(t-1)j(t-1))$$

where $j(t) = e^{t+1} - (t+1)$. Then $\mathcal{L}(j)(s) = \mathcal{L}(ee^t - t - 1) = e\frac{1}{s-1} - \frac{1}{s^2} + \frac{1}{s}$. Therefore

$$\mathcal{L}((e^t - 1)u(t-1)) = e^{-s}\mathcal{L}(j) = e^{-s}(\frac{e}{s-1} - \frac{1}{s^2} + \frac{1}{s})$$

(b)

$$\begin{aligned}\mathcal{L}(y'' + r4y' - 3y) &= s^2\mathcal{L}(y) - (sy'(0) + y(0)) + 4(s\mathcal{L}(y) - y(0)) - 3\mathcal{L}(y) \\ &= (s^2 + 4s - 3)\mathcal{L}(y) - (s+2) - 8 \\ &= (s^2 + 4s - 3)\mathcal{L}(y) - s - 10\end{aligned}$$

So the equation is

$$(s^2 + 4s - 3)Y(s) - s - 10 = e^{-s}(\frac{e}{s-1} - \frac{1}{s^2} + \frac{1}{s}).$$

Problem 5

Consider the following MATLAB commands.

```
syms t s Y;
f=['t^2+exp(t)'];
eq=sym(['D(D(y))(t)+3*D(y)(t)+2*y(t)= ' f]);
leq=laplace(eq,t,s)
algeqn=subs(leq,{'laplace(y(t),t,s)','y(0)','D(y)(0)'},{Y,2,-1});
ytrans=simplify(solve(algeqn,Y));
y=ilaplace(ytrans,s,t)
```

(a) What is the initial value problem that is being solved?
(b) What is the value of the `algeqn` variable? Hint: it is of the form

$$\mathtt{2*Y} + \cdots \mathtt{==} \mathtt{1/(s-1)} + \dots$$

and you should fill in the missing terms.

Solution
(a) The differential equation is

$$y'' + 3y' + 2y = t^2 + e^t \quad y(0) = 2, y'(0) = -1$$

(b)

Take the Laplace transform of both sides:

$$\mathcal{L}(y'') + 3\mathcal{L}(y') + 2\mathcal{L}(y) = \mathcal{L}(t^2) + L(e^t)$$

this gives

$$s^2\mathcal{L}(y) - (sy(0) + y'(0)) + 3(s\mathcal{L}(y) - y(0)) + 2\mathcal{L}(y) = \frac{2}{s^3} + \frac{1}{s-1}$$

(the terms on the right are from the table of Laplace transforms). Plugging in $y(0) = 2, y'(0) = -1$ gives

$$s^2\mathcal{L}(y) - (2s - 1) + 3(s\mathcal{L}(y) - 2) + 2\mathcal{L}(y) = \frac{2}{s^3} + \frac{1}{s-1}$$

and setting $\mathcal{L}(y) = Y$ this is

$$(s^2 + 3s + 2)Y - (2s + 5) = \frac{2}{s^3} + \frac{1}{s-1}$$

Here is the literal output from Matlab:

```
algeqn =

2*Y - 2*s + 3*Y*s + Y*s^2 - 5 == 1/(s - 1) + 2/s^3
```

Table of Laplace Transforms

$\mathcal{L}[t^n](s)$	$\frac{n!}{s^{n+1}}$	for $s > 0$
$\mathcal{L}[t^n e^{at}](s)$	$\frac{n!}{(s-a)^{n+1}}$	for $s > a$
$\mathcal{L}[e^{at}\cos(bt)](s)$	$\frac{s-a}{(s-a)^2+b^2}$	for $s > a$
$\mathcal{L}[e^{at}\sin(bt)](s)$	$\frac{b}{(s-a)^2+b^2}$	for $s > a$
$\mathcal{L}[t^n j(t)](s)$	$(-1)^n J^{(n)}(s)$	where $J(s) = \mathcal{L}[j(t)](s)$
$\mathcal{L}[e^{at} j(t)](s)$	$J(s-a)$	where $J(s) = \mathcal{L}[j(t)](s)$
$\mathcal{L}[u(t-c)j(t-c)](s)$	$e^{-cs}J(s)$	where $J(s) = \mathcal{L}[j(t)](s)$ u is the unit step function