**Math 340, Jeffrey Adams SOLUTIONS**
Final Exam, December 15 , 2010
All Questions 20 points

Question 1 (a) The cross product is $(1,1,2)\times(1,2,2) = (2-4,-2+2,2-1) = (-2,0,1)$. The answer is all multiples of this.

(b) The determinant is $c-1+3$, which equals 0 if and only if $c=-2$. So it is invertible if and only if $c \neq -2$.

(c) For what values of $a$ are $\vec{v} = (5,a,2a)$ and $\vec{w} = (-1+2a, 6-a, 3+a)$ linearly independent? The easiest way is to compute $\vec{v}\times\vec{w}$, and see if it is $\vec{0}$. The cross product is $(a(3+a)-2a(6-a), 5(3+1)-2a(-1+2a), 5(6-1)-a(-a+2a)$. The first coordinate gives $3a^2-9a=0$, i.e. $a=0$ or $a=3$. If $a=0$ the vectors are $(5,0,0)$ and $(-1,6,3)$ which are obviously linearly independent. If $a=3$ they are $(5,3,6)$ and $(5,3,6)$ which are linearly dependent. So they are linearly independent if and only if $a\neq 3$.

(2) Let $M = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 3 & 1 & -1 \end{pmatrix}$.

(a) Using row operations you can get to $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ which has null space $(x,y,z)$ with $y+2z=0$ and $x-z=0$, i.e. spanned by $(1,-2,1)$.

(b) The image is spanned by the columns $(2,1,3),(1,1,1)$ and $(0,1,-1)$. These aren't linearly independent, although any two of them are, and serve as a basis (of course there are infinitely many other possibilities).

(3) Let $f(x,y) = x^2+y^2-xy$.

(a) $\nabla f = (2x-y, 2y-x)$.

(b) This is the tangent approximation: $T(x,y) = (3,1)+\nabla f(3,1)(x-3,y-1) = (3,1)+(5,-1)\cdot(x-3,y-1)$, or $T(x,y) = (3x-6,y)$.

(4) On the interior: $\nabla f(x, y) = \vec{0}$ only at $(\frac{1}{2}, 0)$.

On the boundary

$$(2x - 1, -4y) + \lambda(2x, 2y) = (0, 0)$$

and $x^2 + y^2 = 1$ gives $y = 0, x = \pm 1$ or $\lambda = 2$, $x = \frac{1}{6}$ and $y = \pm\frac{\sqrt{35}}{6}$. Checking these points the maximum is 2 at $(1, 0)$, and the minimum is $-\frac{75}{36}$, at $(\frac{1}{6}, \frac{\sqrt{35}}{6})$.

(5) Make a bowl by rotating the curve $y = 2x^2$ around the $y$-axis, and cutting it off at $y = 4$. Write down an integral to compute the surface are of the bowl. It is *not* necessary to evaluate the integral.

Turn it sideways, and use $x = 2y^2$, or $f(x) = y = \frac{1}{\sqrt{2}}\sqrt{x}$, Then $f'(x) = \frac{1}{2\sqrt{2}\sqrt{x}}$, and

$$\int_0^4 2\pi \frac{1}{\sqrt{2}}\sqrt{x}\sqrt{1 + \frac{1}{8x}}$$

(6) (a) $\vec{x}'(t) = (-\sin(t), \cos(t), 3)$

(b) $\vec{T}(t) = (-\sin(t), \cos(t), 3)/\sqrt{\sin^2(t) + \cos^2(t) + 9} = (-\sin(t), \cos(t), 3)/\sqrt{10}$.

(c) $\vec{T}'(t) = (-\cos(t), -\sin(t), 0)$, so $\kappa(t) = 1/10$ (the constant function).

(7) Let $\vec{F}(x, y, z) = (2x^2, 3xy, xyz)$. Compute $\int_C \vec{F} \cdot d\vec{x}$ where $C$ is the curve $\vec{x}(t) = (t, t^2, t^3)$ for $0 \leq t \leq 2$.

$$\begin{aligned}\int_0^2 (2t^2, 3t^3, t^6) \cdot (1, 2t, 3t^2)\, dt &= \int_0^2 (2t^2 + 6t^4 + 3t^8)\, dt \\ &= [\frac{2}{3}t^3 + \frac{6}{5}t^5 + \frac{3}{9}t^9]_0^2 \\ &= \frac{16}{3} + \frac{192}{5} + \frac{412}{3} = \frac{1072}{5}\end{aligned}$$

(8) Let's hope that $F$ is a gradient vector field. Try $f(x, y)$ such that $f_x = y + \frac{1}{y}$, i.e. $f(x, y) = xy + \frac{x}{y}$. Then $f_x = y + \frac{1}{y}$, and also $f_y = x - \frac{x}{y^2}$. So this works, $\vec{F} = \nabla f$, and therefore the integral is independent of path. Furthermore it is equal to $f(2, 2) - f(1, 1) = 3$.

(9)

$$\begin{aligned}
\int_{-1}^{1}\int_{2x^2}^{1+x^2} (2x + ey)\, dy\, dx &= \int_{-1}^{1} [2xy + \frac{3}{2}y^2]_{2x^2}^{1+x^2}\, dx \\
&= \int_{-1}^{1} (2x + 2x^3 + \frac{3}{2}(1 + x^4 + 2x^2) - 4x^3 - 6x^4\, dx \\
&= [-\frac{9}{2}\frac{1}{5}x^5 - \frac{2}{4}x^4 = x^3 + x^2 + \frac{3}{2}x]_{-1}^{1} \\
&= 2(-\frac{9}{10} + 1 + \frac{3}{2}) = \frac{16}{5}
\end{aligned}$$

(10) Let $T(u, v) = (u^2 - v^2, 2uv)$.

(a) The Jacobian is

$$\det\begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix} = 4(u^2 + v^2)$$

(b) $T$ takes the $0 \le v \le 1$ to $0 \le x \le 1$, and $u = 1, 0 \le v \le 1$ to the parabola $y^2 = 4 - 4x$, etc.

(c)

$$\begin{aligned}
\int_0^1\int_0^1 4(u^2 + v^2)\, du\, dv &= \int_0^1 [\frac{4}{3}u_4^3 uv^2]_0^1\, dv \\
&= \int_0^1 (\frac{+}{4}v^2)\, dv \\
&= [\frac{4}{3}v + \frac{4}{3}v^3]_0^1 = \frac{8}{3}
\end{aligned}$$