Math 403, Jeffrey Adams

Final, May 14, 2010 Take Home

Honor system: open book, but no online or other extensive research

Due 2 PM Tuesday, May 18. If I'm not in my office put it under the door, or in my mailbox (mailroom is on the first floor).

1. Suppose G is a group, containing exactly one nontrivial proper subgroup H (i.e. $H \neq \{e\}$ and $H \neq G$). Show that $G = \mathbb{Z}_{p^2}$ (the cyclic group of order p^2) where p is prime.

2. Let G be the set of 2×2 real matrices of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$.

   (a) Show that G, with matrix multiplication, is a group.

   (b) Show that the set H of matrices of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ is a subgroup of G.

   (c) Show that $H \simeq (\mathbb{R}, +)$, the real numbers with addition (here H is a group under matrix multiplication).

   (d) Show that H is a normal subgroup of G.

   (e) Show that $G/H \simeq \mathbb{R}^* \times \mathbb{R}^*$ where $\mathbb{R}^*$ is the group of non-zero real numbers (with group operation multiplication).

3. Let $G = S_5$, the symmetric group on 6 letters.

   (a) Show that G has 24 elements of order 5.

   (b) If $h \in G$ has order 5, show that $\langle h \rangle$ consists has 5 elements, the identity and 4 elements of order 5.

   (c) Show that G has exactly 6 cyclic subgroups of order 5. Label these $H_1, \dots, H_6$.

   (d) If $g \in G$, $1 \leq i \leq 6$ show that $gH_ig^{-1} = H_j$ for some j.

   (e) For $g \in G$, define a permutation of 6 elements, using part (c): if $gH_ig^{-1} = H_j$ define $g \cdot i = j$. Show that this defines a group homomorphism $G = S_5 \to S_6$.

4. (a) Suppose $\mathbb{F}$ is a field and $\phi : \mathbb{F} \to \mathbb{F}$ is a field homomorphism. Show that $\phi(1) = 0$ or 1.

   (b) Find all field homomorphism from $\mathbb{Q}$ to $\mathbb{Q}$.

5. (a) Determine all group homomorphisms from S_n to $\mathbb{Z}_2$ for $n = 2, 3, 4$.

   (b) For $n \geq 5$, determine all group homomorphisms from S_n to $\mathbb{Z}_2$.

6. Prove that $\mathbb{F} = \mathbb{Q}(\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}, \sqrt[5]{2}, \dots)$ is an algebraic extension of $\mathbb{Q}$, but not a finite extension of $\mathbb{Q}$.

7. Consider the field $\mathbb{F}_{11}$ with 11 elements.

   (a) Show that the polynomial $x^2 - 2$ is irreducible over $\mathbb{F}_{11}$.

   (b) Show that $x^2 - 7$ is irreducible over $\mathbb{F}_{11}$.

   (c) Show that $\mathbb{F}_{11}[x]/\langle x^2 - 2 \rangle$ is isomorphic to $\mathbb{F}_{11}[x]/\langle x^2 - 7 \rangle$.

   (d) Find an element $a + b\sqrt{2} \in \mathbb{F}_{11}[\sqrt{2}]$ $(a, b \in \mathbb{F}_{11})$ satisfying $(a + b\sqrt{2})^2 = 7$.