Math 430, Jeffrey Adams

Homework 8 Solutions

pg. 366, #2,3,4,5; pg. 378, #8,9. Extra credit: pg. 379, #14, 23, 30.

page 366

#2 First of all $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Each of these containments is of degree 2, so $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 4$.

Secondly $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2} + \sqrt{3}) \subset \mathbb{Q}(\sqrt{2}, \sqrt{3})$. From the preceding both containments are degree 2, or $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})]$. Obviously $[\mathbb{Q}(\sqrt{2} + \sqrt{3}) : \mathbb{Q}] \neq 2$, since $\sqrt{2} + \sqrt{3}$ doesn't satisfy a quadratic equation. So the second possibility doesn't occur, which proves the claim.

#3 The splitting field F contains $1, \zeta, \zeta^3$ where $\zeta = e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Then $\zeta^2 = \overline{\zeta} = -\frac{1}{2} - \imath\frac{\sqrt{3}}{2}$. So $F = \mathbb{Q}(i\sqrt{3})$.

#4 The roots are $e^{\pm\pi i/4}, e^{\pm 3\pi i/4}$, i.e. $\pm\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2}$. So the splitting field is $\mathbb{Q}(i\sqrt{2}, \sqrt{2}) = \mathbb{Q}(\sqrt{2}, i)$.

#5 The roots of the two polynomials are $\pm\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. So the splitting field is $\mathbb{Q}(i\sqrt{3})$.

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#8 We have $\mathbb{Q} \subset \mathbb{Q}(\sqrt{15}) \subset \mathbb{Q}(\sqrt{3} + \sqrt{5})$. As in Problem 2, page 366, $\mathbb{Q}(\sqrt{3} + \sqrt{5}) = \mathbb{Q}(\sqrt{3}, \sqrt{5})$, and this has degree 4 over $\mathbb{Q}$. So $[\mathbb{Q}(\sqrt{3} + \sqrt{5}) : \mathbb{Q}(\sqrt{15})] = 2$. For a basis take $\{1, a\}$ where a is any element of $\mathbb{Q}(\sqrt{3} + \sqrt{5}) - \mathbb{Q}(\sqrt{15})$, for example $\{1, \sqrt{3}\}$.

#9 Since $F \subset F(a) \subset E$ and $[E : F] = p$ we have $[E : F(a)][F(a) : F] = p$. Since p is prime $[E : F(a)] = 1$ or p, and $[F(a) : F] = p$ or 1.

extra credit

#14 Let $\alpha = \sqrt{-3} + \sqrt{2}$. Then $\alpha^2 = -3 + 2 + 2\sqrt{-6} = -1 + 2\sqrt{-6}$, and $\alpha^4 = 1 + 4(-6) - 4\sqrt{-6} = -23 - 4\sqrt{-6}$. Then $\alpha^4 + 2\alpha^2 = -23 - 4\sqrt{-6} - 2 + 4\sqrt{-6} = -25$. So $\alpha^4 + 2\alpha^2 + 25 = 0$. We claim $f(x) = x^4 + 2x^2 + 25$ is the minimal polynomial. If not, this polynomial factors. It has no rational roots, so the only possibility is two quadratic factors. But $\sqrt{-3} + \sqrt{2}$ is not the solution to a quadratic equation.

#23 The polynomial factors into $a(x - \alpha)(x - \beta)$, where $\alpha, \beta = (-b \pm \sqrt{b^2 - 4ac}/2a)$. Take $\alpha = \sqrt{b^2 - 4ac}$. Note that the splitting field is $\mathbb{Q}$ if and only if $b^2 - 4ac \in \mathbb{Q}^2$.

#30 This is not true in general. For example take $n = 2, a = 1$: $[\mathbb{Q}(\sqrt{1}) : \mathbb{Q}] = 1 < 2$, for goodness sake.