Math 463, Fall 03 FINAL
Open Book

Question 1 [25 points]

Write $-8i = 8(-i) = 8e^{-\frac{\pi}{2}i}$. Then if $z = re^{i\theta}$ we have to solve

$$r^3 e^{3i\theta} = 8e^{-\frac{\pi}{2}i}$$

which gives $r = 2$ and $3\theta = -\frac{\pi}{2} + 2\pi k$ for $k \in \mathbb{Z}$. Therefore $\theta = -\frac{\pi}{6} + \frac{2\pi}{3}ki$. Then $k = 0$ gives $\theta = -\frac{\pi}{6}$ and so $z = 2e^{-\frac{\pi}{6}i} = 2(\cos(-\frac{\pi}{6}) + i\cos(-\frac{\pi}{6})) = 2(\frac{\sqrt{3}}{2}) - 2\frac{1}{2}i = \sqrt{3} - i$. Similarly $k = 1$ gives $\theta = -\frac{\pi}{6} + \frac{2\pi}{3}i = \frac{\pi}{2}i$, and $z = 2e^{i\frac{\pi}{2}} = 2i$. Finally $k = 2$ gives $\frac{7\pi}{6}$ and $z = 2 - \frac{\sqrt{3}}{2} + 2i(-\frac{1}{2}) = -\sqrt{3} - i$.

Question 2 [25] Let $z(t) = 3e^{int}$ with $0 \le t \le 2\pi$. Then $z'(t) = 3ie^{it}$ and the integral becomes

$$\begin{aligned}\int_0^{2\pi} \overline{(3e^{it})^3} 3ie^{it}\,dt &= \int_0^{2\pi} \overline{27e^{-3it}} 3ie^{it}\,dt \\ &= 81\int_0^{2\pi} e^{2it}\,dt \\ &= 81(e^{4\pi i} - e^0) = 0\end{aligned}$$

Question 3 [40]

(a)Write $\frac{1}{z} = \frac{1}{i+(z-i)} = \frac{-1}{-i-(z-i)} = \frac{-1}{-i(1-i(z-i))} = \frac{-i}{1-i(z-i)}$. Then the expansion $\frac{1}{1-z} = \sum_0^\infty z^n$ ($|z| < 1$) gives

$$\frac{1}{z} = -i\sum_0^\infty [i(z-i)]^n = -i\sum_0^\infty i^n(z-i)^n = -\sum_0^\infty i^{n+1}(z-i)^n = \sum_0^\infty i^{n-1}(z-i)^n$$

This is valid for $|z - i| < 1$.

(b) Since $\frac{d}{dz}Log(z) = \frac{1}{z}$, $Log(z)$ is an antiderivative of $\frac{1}{z}$, and we can integrate the result from (a) to give

$$Log(z) = \sum_0^\infty i^{n-1}\frac{1}{n+1}(z-i)^{n+1} + C$$

for some C.

Note the C: when we do the indefinite integral (antiderivative) we don't know the constant. Note that the sum, evaluated at i gives 0, which gives $Log(i) = 0 + C$, or $C = i\frac{\pi}{2}$.

(c)

If you are looking for convergence, the answer is as long as the circle doesn't contain 0, i.e. $|z - z_0| < |z_0|$.

If you want to know where it converges to the function, you can take the largest circle for which $Log(z)$ is analytic. The branch cut is along the negative real axis. So if $z + 0 = x + iy$ and $x \ge 0$ the circle will go through 0, and has radius $|z| = \sqrt{x^2 + y^2}$, i.e. $|z - z_0| < |z_0|$. If $x < 0$ then the circle will hit the negative axis, and has radius y, so $|z - z_0| < y$.

(d) The function $Log(z)$ is not analytic in any deleted neighborhood of 0 (i.e.

Question 4 [25] Write $f(z) = \sum_0^\infty a_n z^n$. Fix $N \geq 2$. Then

$$0 = \int_C \frac{f(z)}{z^N}\, dz = \sum_0^\infty a_n \int \frac{z^n}{z^N}\, dz$$
$$= \sum_0^\infty a_n \int z^{n-N}\, dz$$

Each integrand is 0 unless $n - N = -1$, i.e. $n = N - 1$ which gives

$$0 = 2\pi i a_{N-1}$$

so $a_{N-1} = 0$ for all $N \geq 2$, i.e. $a_n = 0$ for $n \geq 1$. Therefore the only non–zero coefficient is a_0, and $f(z) = a_0$, a constant function.

Question 5 [30]

(a) At $z_0 = 0$ write $f(z) = \frac{e^{z-1}/(z-1)^2}{z-1}$, which gives a residue of $\frac{e^{-1}}{(-1)^2} = e^{-1}$.

At $z_0 = 1$ write $f(z) = \frac{e^{z-1}/z}{(z-1)^2} = \frac{\phi(z)}{(z-1)^2}$. The residue is $\phi'(1)$; $\phi'(z) = \frac{e^{z-1}z - e^{z-1}}{z^2}$, and plugging in 1 gives $\frac{e^0 - e^0}{1} = 0$.

(b) Suppose the circle goes around 0 k times. Then the integral is $2\pi i k e^{-1}$.

Question 6 30] Write $\cos(z) = Re(e^{ix})$ and compute $\int_{-\infty}^{\infty} \dfrac{e^{ix}}{(x^2+9)^2}\, dx$. Write this as a complex integral over the contour C_R on the real axis as usual. Then complete this contour to a closed contour by adding a semicircle in the upper half-plane D_R. Then

$$\int_{C_R} = \int_{C_R + D_R} - \int_{D_R}$$

By the usual argument $\lim_{R\to\infty} \int_{D_R} = 0$, so we may compute the given integral by residues. That is we compute

$$\int_{C_R + D_R} \frac{e^{iz}}{(z^2+9)^2}\, dz$$

The poles are at $\pm 3i$. The residue at $3i$ is computed by

$$\frac{e^{iz}}{(z^2+9)^2} = \frac{e^{iz}}{(z+3i)^2(z-3i)^2}$$
$$= \frac{e^{iz}/(z+3i)^2}{(z-3i)^2}$$
$$= \frac{\phi(z)}{(z-3i)^2}$$

and we need to compute $\phi'(3i)$. That is $\phi'(z) = \frac{ie^{iz}(z+3i)^2 - e^{iz}2(z+3i)}{(z+3i)^4}$ and

$$\begin{aligned}
\phi'(3i) = \frac{ie^{-3}(6i)^2 - e^{-3}2(6i)}{(6i)^4} &= \frac{e^{-3}(6i)(i(6i)-2)}{6^4(-1)} \\
&= \frac{e^{-3}(6i)(-8)}{6^4(-1)} \\
&= \frac{-ie^{-3}(8)}{6^3} \\
&= \frac{-ie^{-3}(8)}{2^3 3^3} \\
&= \frac{-ie^{-3}}{27}
\end{aligned}$$

Therefore the integral is $2\pi i\frac{-ie^{-3}}{27} = \frac{2}{27}\pi e^{-3}$

Question 7 [25] Since $1 \to 0$ and $-1 \to \infty$ and circles go to circles, the image is a line through the origin. Since $i \to \alpha\frac{i-1}{i+1}$ it is the line through the origin and this point.

The disk goes to a half plane with boundary this line; the one containing the image of 0, i.e. $-\alpha$.