Math 463, Jeffrey Adams

Solutions to selected problems in Homework set 10

Assignment 10, due Monday, December 8: page 78, #1c,d, 3; page 350, #1,3; page 305, #2,14. I graded everything but page 350 #1, for a total of 35 points.

1. page 78, #1c Here $v(x,y)=\cosh(x)\cos(y)$. Note that by formula (10), page 106, $u+iv=\sinh(x)\sin(x)-i\cosh(x)\cos(y)=-i(\cosh(x)\cos(y)+i\sinh(x)\sin(y))=-i\cosh(z)$.

2. #1d. In this case $v(x,y)=\frac{x}{x^2+y^2}$, and $u+iv=\frac{1}{x^2+y^2}(y+ix)=i\frac{1}{x^2+y^2}(x-iy)=i\frac{1}{z\overline{z}}\overline{z}=\frac{i}{z}$.

3. #3 If v is a harmonic conjugate of u then $u_x=v_y$. If u is a harmonic conjugate of v then $v_y=-u_x$. Together these imply $u_x=v_y=-u_x$, so $u_x=0$, and u doesn't depend on x. Similarly $u_y=-v_x=u_y$ implies it doesn't depend on y either: u is constant, and so is v.

4. page 350, #1 If $f(z)=z^2$ then $f'(z)=2z$ and $f'(2+i)=4+2i$. Therefore the angle of rotation is $arg(4+2i)=\tan^{-1}(\frac{2}{4})$, and the scale factor is $|4+2i|=\sqrt{16+4}=2\sqrt{5}$.

5. #3 The line $y=x-1$ goes through $(1,0),(0,-1)$, and ∞, i.e. $1,-i$ and ∞. The image of these points under $z\to\frac{1}{z}$ are $1,i$ and 0 respectively. Therefore the image of this line is the circle containing these three points: it is centered at $\frac{1}{2}+\frac{1}{2}i$. It is clear from a picture that the tangent at 1 has angle $\frac{\pi}{4}$.

   Explicitly, parametrize the curve by $z(t)=\frac{1}{2}+i\frac{1}{2}+\frac{\sqrt{2}}{2}e^{it})$. Then $z'(t)=\frac{\sqrt{2}}{2}ie^{it}$, and set $t=-\frac{\pi}{4}$ to see $z'(t)=i\frac{\sqrt{2}}{2}e^{-\frac{\pi i}{4}}=\frac{\sqrt{2}}{2}e^{\frac{i\pi}{2}}e^{-\frac{i\pi}{4}}=\frac{\sqrt{2}}{2}e^{\frac{\pi i}{4}}$. This says the angle is $\frac{\pi}{4}$.

   The real axis goes to itself. Then line $y=x-1$ makes an angle of $\frac{\pi}{4}$ with the real axis, and the same holds for the circle. This confirms conformality for these curves.

6. page 305, #2 The line $x=c_1$ goes to a circle, symmetric in the x axis, containing $\frac{1}{c_1}$ and $\frac{1}{\infty}=0$. This is the circle of radius $\frac{1}{2c_1}$ centered at $-c_1$ and containing 0. The region to the left of the line goes to the interior of the circle. The region to the right goes to the exterior; in particular 0 goes to ∞.

   In the limit $c_1=0$ we get a circle of infinite radius, which is the imaginary axis. The imaginary axis goes to itself, and the "interior" of this circle is the left half plane, which goes to itself.

7. page 305, #14

   (a) If $z=x+iy$ then $x^2+y^2=z\overline{z}$ and $Bx+Cy=\frac{1}{2}B(z+\overline{z})+\frac{1}{2i}B(z-\overline{z})=\frac{1}{2}(B-Ci)z+\frac{1}{2}(B+Ci)\overline{z}$. Inserting these and multiplying by 2 gives the result.

   (a) Plug in $w=\frac{1}{z}$ to get $2A\frac{1}{w\overline{w}}+(B-Ci)\frac{1}{w}+(B+Ci)\frac{1}{\overline{w}}+2D=0$. Multiply by $w\overline{w}$ to get $2A+(B-Ci)\overline{w}+(B+Ci)w+2Dw\overline{w}=0$. The same idea used in (a) gives (4).