Math 463, Jeffrey Adams

Solutions to selected problems in Homework set 2

- page 21, #7. You need to use the fact that $Re(z_1), Re(z_2) > 0$.

  We always have $arg(z_1z_2) = arg(z_1) + arg(z_2)$. Therefore

  $$Arg(z_1z_2) = Arg(z_1) + Arg(z_2) + 2\pi k \tag{1}$$

  for some k. We have to show $k = 0$.

  Note that $Re(z) > 0$ if and only if $-\frac{\pi}{2} < Arg(z) < \frac{\pi}{2}$. So $-\frac{\pi}{2} < Arg(z_1), Arg(z_2) < \frac{\pi}{2}$. Therefore $-\pi < Arg(z_1) + Arg(z_2) < \pi$. Since $-\pi < Arg(z_1z_2) < \pi$ by (1) we conclude $k = 0$.

- #9. This is an if and only if statement, so there are two directions to show.

  Note that $z_1 = c_1c_2$ and $z_2 = c_1\overline{c_2}$ implies $|z_1| = |c_1c_2| = |c_1||c_2| = |c_1||\overline{c_2}| = |z_2|$. This proves one direction.

  On the other hand if $|z_1| = |z_2|$ write $z_i = re^{i\theta_i}$. Then take $c_1 = \sqrt{r}e^{i\frac{\theta_1+\theta_2}{2}}$ and $c_2 = \sqrt{r}e^{i\frac{\theta_1-\theta_2}{2}}$.

- #10. The first part of the identity is simply $(1-z)(1+z+\dots z^n) = (1 - z + z - z^2 + z^2 + \cdots - z^{n+1}) = 1 - z^{n+1}$.

  For the second take $z = e^{i\theta}$, plug in to the formula, and take the real part. The left hand side gives $1 + \cos(\theta) + \cdots + \cos(n\theta)$. The right hand side is $Re(\frac{1-e^{i\theta(n+1)}}{1-e^{i\theta}})$. It is enough to show this equals the right hand side of the assertion, i.e. it is enough to show

  $$Re(\frac{1-e^{i\theta(n+1)}}{1-e^{i\theta}}) = \frac{1}{2} + \frac{\sin((2n+1)\theta/2)}{2\sin(\theta/2)}$$

  Write

  $$\begin{aligned} 1 - e^{i\theta} &= e^{i\theta/2}(e^{-i\theta/2} - e^{i\theta/2}) = e^{i\theta/2}(cos(\theta/2) - i\sin(\theta/2) - \cos(\theta/2) - i\sin(\theta/2)) \\ &= e^{i\theta/2}(-2isin(\theta/2) \end{aligned}$$

Then

$$
\begin{aligned}
Re(\frac{1-e^{i\theta(n+1)}}{1-e^{i\theta}}) &= Re(\frac{1-e^{i\theta(n+1)}}{-e^{i\theta/2}2i\sin(\theta/2)}) \\
&= Re(\frac{e^{-i\theta/2}(1-e^{i\theta(n+1)})}{-2i\sin(\theta/2)}) \\
&= Re(\frac{e^{-i\theta/2}-e^{i\theta(2n+1)/2}}{-2i\sin(\theta/2)}) \\
&= Re(\frac{\cos(\theta/2)-i\sin(\theta/2)-\cos((2n+1)\theta/2)-i\sin((2n+1)\theta/2)}{-2i\sin(\theta/2)} \\
&= \frac{1}{2}+\frac{\sin((2n+1)\theta/2)}{2\sin(\theta/2)}
\end{aligned}
$$

- page 29, #7.

  In the identity $1+z+\cdots+z^m = \frac{1-x^{m+1}}{1-z}$ take $z=c$ and $m=n-1$. Then $m+1=n$ and this gives $1+c+\cdots+c^{n-1} = \frac{1-c^n}{1-z}$. Since $c^n=1$ this equals 0.