Math 463, Jeffrey Adams

Solutions to selected problems in Homework set 3

1. page 53, #9. Note that $|\sin(x)| \le 1$, and $\lim_{x\to 0}\sin(x)$ does not exist.

   So: $|g(z)| \le M$ does not imply $\lim_{z\to z_0} g(z)$ exists. You *cannot* apply Theorem 2, page 47; the assumption that both limits exist doesn't hold.

   So: given ϵ, there exists δ such that $|z - z_0| < \delta$ implies $|f(z) - f(z_0)| < \epsilon\frac{1}{M}$. Then $|z - z_0| < \delta$ imples $|f(z)g(z) - 0| = |f(z)g(z)| = |f(z)||g(z)| \le \epsilon\frac{1}{M}M = \epsilon$. This proves the limit is 0.

2. page 59, #7. It is not enough to take the formula $\frac{d}{dz}z^n = nz^{n-1}$ for $n > 0$ and let $m = -n$. You are trying to prove that this formal procedure is legitimate. You need to use the quotient rule. Suppose $n > 0$:

$$\frac{d}{dz}\frac{1}{z^n} = \frac{0z^n - nz^{n-1}}{z^{2n}} = -nz^{n-1-2n} = -nz^{-n+1}.$$

   Therefore $\frac{d}{dz}z^{-n} = (-n)z^{(-n)+1} = (-n)z^{-((-n)-1)}$. Letting $m = -n$, so $m < 0$ gives $\frac{d}{dz}z^m = mz^{m-1}$.

3. #9. The derivative is easily seen to be $\lim_{z\to 0}\frac{\overline{z}^2}{z^2}$. Take the limit along the ray $z = re^{i\theta}$ with θ fixed. This becomes $\lim_{r\to 0} re^{-2i\theta}re^{i\theta} = \lim_{r\to 0} e^{-3i\theta}$. This can take on any value of absolute value 1, depending on θ, so the limit doesn't exist.

4. page 68, #3.

   (a) (a) Note that $\frac{1}{z} = \frac{1}{z}\frac{\overline{z}}{\overline{z}} = \frac{\overline{z}}{|z|2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$, so $u(x, y) = \frac{x}{x^2+y^2}$ and $v(x, y) = -\frac{y}{x^2+y^2}$.
   This is easier in polar coordinates: $\frac{1}{z} = \frac{1}{r}e^{-i\theta} = \frac{1}{r}\cos(\theta) - i\frac{1}{r}\sin(\theta)$. Therefore $u(r, \theta) = \frac{1}{r}\cos(\theta)$ and $v(r, \theta) = -\frac{1}{r}\sin(\theta)$.

   (b) (b) This is differential if and only if $x = y$.

   (c) (c) Note that $z - \overline{z} = 2iIm(z)$ so $Im(z) = -\frac{i}{2}(z - \overline{z})$ and $zIm(z) = -\frac{i}{2}z(z - \overline{z}) = -\frac{i}{2}z^2 + \frac{i}{2}z\overline{z} = -\frac{i}{2}z^2 + \frac{i}{2}|z|^2$. Since z^2 is analytic for all z the given function is differential exactly where $|z|^2$ is, i.e. 0.

5. #6. Use polar coordinates (even though the exercise asks about u_x, etc., this is equivalent). That is $f(re^{i\theta}) = \frac{r^2e^{-i\theta}}{re^{i\theta}} = re^{-3i\theta} = r\cos(3\theta) - ir\sin(3\theta)$. Therefore $u(r, \theta) = r\cos(3\theta)$ and $v(r, \theta) = -r\sin(3\theta)$. The Cauchy–Riemann equations give $r\cos(3\theta) = -3r\cos(3\theta)$ and $-3r\sin(3\theta) = r\sin(3\theta)$. These hold at 0.