Math 463, Jeffrey Adams

Solutions to selected problems in Homework set 4

Assignment 4, due Friday, October 3: page 89, #7,8 a,c, 11; page 94, # 1,3,5; page 96, #2; page 103, # 2 4, 12 a; page 107, #10

I graded everything (5 points each) but page 94, #3, page 96, #2, for a total of 45 points, plus 5 extra credit for doing page 99, #2b, 3.

1. page 89, #7. Write $z = x+iy$. Then $|e^{-2x}| = |e^{-2x}e^{-2iy}| = |e^{-2x}||e^{-2iy}|$. The second term is 1, and the first is positive, so this equals e^{-2x}. So $|e^{-2x}| < 1$ if and only if $e^{-2x} < 1$. Taking log of both sides gives $-2x < 0$, or $x > 0$.

   Note that $|e^{-2iy}| = 1$ and it is not necessary to expand this into sin an cos.

2. #8. a) Write $-2 = 2e^{i\pi}$, so $\log(-2) = \ln(2) + \pi i + 2\pi ik$ $(k \in \mathbb{Z})$.

3. #11. To say $x \to -\infty$ implicitly means keeping y fixed. So $e^{x+iy} = e^x e^{iy}$ goes to 0 along the ray with angle y.

   On the other hand if $y \to \infty$ with x fixed then $|e^{x+iy}| = e^x$, so these points are all on the circle of radius e^x. The point goes around this circle infinitely often as $y \to \infty$.

4. page 94, #1 a) Write $-ei = e(-i) = ee^{-\pi i/2}$.

   b) Write $1 - i = \sqrt{2}e^{-\pi i/4}$.

5. #5. $log(i^2) = log(-1) = log(e^{i\pi}) = i\pi + 2\pi ik$ $(k \in \mathbb{Z})$.

   On the other hand $2\log(i) = 2\log(e^{i\pi/2}) = 2(i\pi/2) + 2\pi ik = i\pi + 4\pi ik$.

6. page 96, #2 This comes down to $Arg(\theta_1 + \theta_2) = Arg(\theta_1) + Arg(\theta_2) + \{-\pi, 0, \pi\}$. The point is if $-\pi \le \theta_1, \theta_2 \le \pi$, then $-2\pi \le \theta_1 + \theta_2 \le 2\pi$ – therefore the "error" is at most $\pm\pi$.

7. page 103, #2.

   $$\cos(z)+i\sin(z) = \frac{e^{iz}+e^{-iz}}{2} + i\frac{e^{iz}-e^{-iz}}{2i} = \frac{e^{iz}+e^{-iz}}{2} + \frac{e^{iz}-e^{-iz}}{2} = e^{iz}$$

   It is possible to do this by expanding $\cos(z), \sin(z)$ into cos, sin, cosh, sinh of the real and imaginary parts, but this is much more complicated.

8. #4. Take the identity $\sin(z + z_2) = \sin(z)\cos(z_2) + \cos(z)\sin(z_2)$ and differentiate both sides: $\frac{d}{dz}\sin(z+z_2) = \frac{d}{dz}\sin(z)\cos(z_2)+\cos(z)\sin(z_2)$. This gives $\cos(z+z_2) = \cos(z)\cos(z_2)-\sin(z)\sin(z_2)$. Then take $z = z_1$.

9. #12 a)

$$
\begin{aligned}
2\sin(z_1+z_2)\sin(z_1-z_2) &= 2\frac{e^{i(z_1+z_2)}-e^{-i(z_1+z_2)}}{2i}\frac{e^{i(z_1-z_2)}-e^{-i(z_1-z_2)}}{2i} \\
&= \frac{2}{-4}(e^{i(z_1+z_2)}-e^{-i(z_1+z_2)})(e^{i(z_1-z_2)}-e^{-i(z_1-z_2)}) \\
&= -\frac{2}{4}(e^{2iz_1}-e^{2iz_2}-e^{-2iz_1}+e^{-2iz_1}) \\
&= -\frac{2}{2}(\frac{1}{2}(e^{2iz_1}-e^{2iz_2})-\frac{1}{2}(e^{-2iz_1}+e^{-2iz_1})) \\
&= -\cos(2z_2)+\cos(2z_1)
\end{aligned}
$$

10. page 107, #10

$$
\frac{d}{dz}\tanh(z) = \frac{\frac{d}{dz}\sinh(z)\cosh(z)-\sinh(z)\frac{d}{dz}\cosh(z)}{\cosh^2(z)} = \frac{\cosh^2(z)-\sinh^2(z)}{\cosh^2(z)}
$$

Use the fact that $\cosh^2(z)-\sinh^2(z)=1$.