Math 463, Jeffrey Adams

Solutions to selected problems in Homework set 7

Assignmnet 7, due Monday, November 3 page 188, #1,3,6,7,10; page 198, #2,7 I graded everything, 5 points each, for a total of 35.

1. page 188, #6. Once you have $f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{4n+2}$ you should use this to show $f^{(m)}(0) = 0$ for some m. That is $f(z) = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} z^m$, so
$$\sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} z^m = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{4n+2}$$
and we conclude $f^{(m)}(0) = (-1)^n/(2n+1)!$ if $m = 2n+2$, and 0 otherwise. That is $f^{(m)}(0) = 0$ unleess m is of the form $4n+2$, i.e. $m = 2, 6, 10, \ldots$.

2. #10. Since $\tanh(z)$ has a pole at $\pm\frac{\pi}{2}i$ the radius of convergence is $\frac{\pi}{2}$.

   Although $\tanh(z) = \frac{\sinh(z)}{\cosh(z)}$ you *cannot* use the Taylor series of sinh and cosh to get one for tanh. If $\sinh(z) = \sum a_n z^n$ and $\cosh(z) = \sum b_n z^n$ then $\tanh(z) = \frac{\sum a_n z^n}{\sum b_n z^n}$ but this doesn't tell you anything about how to write $\tanh(z) = c_n z^n$. It certainly isn't the case that $c_n = \frac{z_n}{b_n}$ (since when is $\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}$?).

   You just have to compute derivatives of $f(z) = \tanh(z)$: $f'(z) = sech^2(z)$, $f''(z) == 2sech^2(z)\tanh(z)$, $f'''(z) = -4sech(z)\tanh^2(z) - 2sech^4(z)$. At 0 this gives $f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -2$. Therefore $\tanh(z) = z - \frac{2}{3!}z^3 + \cdots = z - \frac{1}{3}z^3 + \ldots$.

3. page 198, #2 You need to find the Taylor expansion of e^z about -1, i.e. as a power series in $z+1$. There is a cute trick here: $e^z = \sum \frac{1}{n!} z^n$, so $e^{z+1} = \sum \frac{1}{n!}(z+1)^n$. For most functions this wouldn't tell you anything about $f(0)$, but $e^{z+1} = ee^z$, so $ee^z = \sum \frac{1}{n!}z^n$, and $e^z = \frac{1}{e}\sum \frac{1}{n!}z^n$. Therefore $\frac{e^z}{(z+1)^2} = \frac{1}{e}\sum \frac{1}{n!}(z+1)^{n-2}$.

4. #6 These are both Laurent series about $z_0 = 0$. The $\frac{1}{z}$ term is fine as it is. You have to expand $\frac{1}{1+z^2}$ as a Taylor Laurent series about 0.

   If $|z| < 1$:
$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty}(-z^2)^n = \sum_{0}^{\infty}(-1)^n z^{2n}.$$
   If $|z| > 1$:
$$\frac{1}{1+z^2} = \frac{1}{z^2}\frac{1-(-\frac{1}{z^2})}{=}\frac{1}{z^2}\sum(-\frac{1}{z^2})^n = \frac{1}{z^2}\sum(-1)^n z^{-2n}$$
   The rest of the problem follows easily from this.