Math 463, Jeffrey Adams

Solutions to selected problems in Homework set 9

Assignment 9, due Monday, November 24: page 280, #1,2,7; page 296, #1,4; 5 points each, for a total of 25

1. page 280, #1. After the substitution you have $\int_C \frac{1}{2z^2+5iz-2}\,dz$. The roots of the denominator are $-\frac{1}{2}i, -2i$, and only $-\frac{1}{2}i$ is inside the contour. The residue is $\frac{1}{3i}$, and the solutions is $2\pi i\frac{1}{3i} = \frac{2\pi}{3}$.

2. #2 The standard substitution gives $-4i\int_C \frac{z}{z^4-6z^2+1}\,dz$. The denominator has roots $z^2 = 3\pm 2\sqrt{2}$, or $z = \pm\sqrt{3\pm 2\sqrt{2}}$. Only $\pm\sqrt{3-2\sqrt{2}}$ are inside the contour. Note that $\sqrt{3-2\sqrt{2}} = 1-\sqrt{2}$.

   These are simple poles. Use Theorem 2, page 243 to evaluate the residues. The derivate of z^4-6z^2+1 is $4z^3-12z = 4z(z^2-3)$, so at $1-\sqrt{2}$ the residue is $\frac{1-\sqrt{2}}{4(z-\sqrt{2})((1-\sqrt{2})^2-3)} = \frac{1}{8\sqrt{2}}$. At $-1+\sqrt{2}$ the residue is the same. So the solution is $2\pi i(-4i)\frac{2}{8\sqrt{2}} = 2\sqrt{2}$.

   Here is a clever trick due to one of the students: use the substitution $z = e^{2i\theta}$. Then $d\theta = \frac{dz}{2iz}$. Also $\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta)) = \frac{1}{2}(1-\frac{z+\frac{1}{z}}{2}) = \frac{2-z+\frac{1}{z}}{4}$. The integral becomes $4i\int_C \frac{1}{z^2-6z+1}\,dz$. Notice this is a quadratic in the denominator, and therefore much easier to compute.

3. #7 I discussed this in class (use the binomial theorem).

4. page 296, #1 The poles are at $s = \pm\sqrt{2}, \pm i\sqrt{2}$. The residues at $\pm\sqrt{2}$ are $\frac{e^{\pm\sqrt{2}t}}{2}$, and at $\pm\sqrt{2}i$ are $\frac{e^{\pm\sqrt{2}it}}{2}$. Adding these we get $\cos(\sqrt{2}t) + \cosh(\sqrt{2}t)$.

5. #4. The poles are at $\pm ai$. Use the technique of Example 1, page 292. Write $\frac{s^2-a^2}{(s^2+a^2)^2} = \frac{(s^2-a^2)/((s+ai)^2)}{(s-ai)^2} = \phi(s)/(s-ai)^2$. The sum of the residues of at $\pm ai$ is $2Re[(e^{ait}(b_1+b_2t)]$ where $\frac{s^2-a^2}{(s^2+a^2)^2} = \frac{b_1}{s-ai} + \frac{b_2t}{(s-ai)^2} + \ldots$. Then $b_1 = \phi'(ai) = 0$, and $b_2 = \phi(ai) = \frac{1}{2}$.

   This gives $2Re(e^{ait}(\frac{1}{2}t)) = t\cos(at)$.