Math 744, Fall 2014
Jeffrey Adams
Homework IV SOLUTIONS

(1) Find an explicit two-to-one map $SL(2,\mathbb{C}) \to SO(3,\mathbb{C})$.

Solution: The adjoint representation. The kernel is $\pm I$.

(2) Find an explicit two-to-one map $SL(4,\mathbb{C}) \to SO(6,\mathbb{C})$.

Solution: We need a 6-dimensional representation of $G = SL(4,\mathbb{C})$. A natural guess is $W = \bigwedge^2(V)$ where $V = \mathbb{C}^4$ is the standard (tautological) module. The group G acts naturally on W: $\pi(g)(v \wedge w) = (gv) \wedge (gw)$. This gives a map from G to $GL(6,\mathbb{C})$. Does it preserve a symmetric bilinear form?

In general, if $V = \mathbb{C}^{2m}$ there is a natural bilinear form on $\bigwedge^m(V)$. Fix an isomorphism $\phi : \bigwedge^{2m}(V) \to \mathbb{C}$, and define

$$(\alpha,\beta) = \phi(\alpha \wedge \beta) \quad (\alpha,\beta \in \bigwedge^m(V)).$$

It is easy to see $(\,,\,)$ is bilinear and symmetric if m is even, or skew-symmetric if m is odd.

Applying this to W gives a symmetric bilinear form, which is easily seen to be preserved by G. The kernel is $\pm I$.

(3) Find an explicit two-to-one map $Sp(4,\mathbb{C}) \to SO(5,\mathbb{C})$.

Solution: We need a 5-dimensional representation of $G = Sp(4,\mathbb{C})$. Start with $W = \bigwedge^2(V)$ where $V = \mathbb{C}^4$ is the standard module, equipped with a symplectic form. Then $\dim(W) = 6$.

Suppose $V = \mathbb{C}^n$, with a symplectic form $\langle\,,\rangle$. Then there is a natural map $\phi : \bigwedge^k(V) \to \bigwedge^k(V^*) = \bigwedge^k(V)^*$ (where V^* is the dual). This is given by

$$\phi(v_1 \wedge \cdots \wedge v_k)(w_1 \wedge \cdots \wedge w_k) = \det(\{\langle v_i, w_j\rangle\}_{i,j})$$

(determinant of this $n \times n$ matrix). This defines a form

$$(\alpha,\beta) = \phi(\alpha)(\beta)$$

which is symplectic if k is odd, and symmetric if k is even.

Applying this, we have a map $Sp(4,\mathbb{C}) \to SO(6,\mathbb{C})$. But we want $SO(5,\mathbb{C})$ instead.

Back to $V = \mathbb{C}^n$, fix a symplectic basis $\{e_1,\dots,e_n,f_1,\dots,f_2\}$ of V: $\langle e_i, e_j\rangle = \langle f_i, f_j\rangle = 0$ and $\langle e_i, f_i\rangle = \delta_{i,j}$. Define

$$\tau = \sum_{i=1}^n e_i \wedge f_i$$

Using the fact that

$$v = \sum_{i=1}^n [\langle v, f_i\rangle e_i - \langle v, e_i\rangle f_i]$$

it is easy to see that

$$\langle v, w\rangle = (v \wedge w, \tau) \quad (v, w \in V)$$

It follows that $Sp(4)$ fixes $\tau \in W$, and therefore its orthogonal complement, which has dimension 5. The kernel is $\pm I$.

(4) Prove the following result. Suppose V is a vector space and $R \subset V$ is a finite subset which spans V. If $\alpha \neq 0 \in R$ there exists at most one pseudo-reflection s such that $sv = -v$ and $s(R) = R$. (Recall a pseudo-reflection is any linear map satisfying $sv = v$ for all v in a subspace of codimension 1, and $sw = -w$ for some w).

Hint: Suppose s, s' both satisfy the condition, so if $t = ss'$ then $t\alpha = \alpha$ and $tv = v + f(v)\alpha$ for some $f \in V^*$. Consider powers of t.

This Lemma says that a root system can be defined without use of a bilinear form: the map $R \ni \alpha \to \alpha^\vee \in V^*$ is uniquely determined.

Solution: Suppose s, s' both satisfying the condition, and let $t = ss'$. Then $t(\alpha) = \alpha$. Also for any v, $s(v) - v \in \mathbb{C}\langle\alpha\rangle$, and similarly s'. Therefore

$$t(v) = v + f(v)\alpha$$

for some $f : V \to \mathbb{C}$. Since t is linear it is clear that f is linear. It is easy to see that by induction we have

$$t^n(v) = v + nf(v)$$

for all $n \geq 0$.

Since t is an automorphism of the finite set R, it has some finite order m. Taking $n = m$ we have

$$t^m(v) = v + mf(v) = v$$

for all v. Therefore $f(v) = 0$ for all v, $t(v) = v$ and $s = s'$.

(5) If R is an irreducible root system, and $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ is a set of simple roots, then R has a unique maximal root β (i.e. $\alpha > 0$ implies $\beta + \alpha \notin R$).

Set $\alpha_0 = -\beta$, and define integers a_i by $a_0 = 1$ and

$$\sum_{i=0}^{n} a_i\alpha_i = 0.$$

Let $\widehat{\Pi} = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$. Define the extended Dynkin diagram in the same way as the ordinary Dynkin diagram, applied to $\widehat{\Pi}$. Label each node $0 \leq i \leq n$ of the extended diagram with a_i.

(a) Draw the extended Dynkin diagrams for the classical groups, including the labels.

(b) Suppose R is simply laced. Show that a_i is one-half the sum of the labels on all adjacent nodes.

Solution: Using $\sum_{i=0}^{n} a_i\alpha_i = 0$, compute, for any j:

$$\langle \sum_{i=0}^{n} a_k\alpha_i, {}^{\vee}\alpha_j \rangle = 0$$

The term $k = i$ on the left gives 2, and all terms adjacent to the k^{th} node give -1, so $2 = \sum a_r$ where the sum runs over the adjacent nodes.

(c) The extended Dynkin diagram of type E_8 has α_0 adjacent only to the end of the long arm (with bond 1). Use (b) to compute the labels. Show that $\sum_{i=0}^{n} a_i = 30$.

Solution: $1+2+3+4+5+6+3+4+2 = 30$.

(d) For a classical group show that the number of nodes of the extended diagram labelled 1 is the order of the center of the simply connected group.

Solution: See (a).

(6) The root system of type D_4 has an outer automorphism of order 3 which preserves a set of positive roots (corresponding to an automorphism of the Dynkin diagram). Write down this automorphism explicitly.

Solution: The automorphism cyclically permutes $\{e_1 - e_2, e_3 - e_4, e_3 + e_4\}$ and fixes $e_2 - e_3$. A little linear algebra says that, in the usual coordinates $e_1, \dots, e_4$, this is given by the matrix

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

For example $A(1,-1,0,0)^t = (0,0,1,-1)^t$ and $A(0,1,-1,0)^t = (0,1,-1,0)^t$.

Note that $A(3,2,1,0)=(3,2,1,0)$. This is because $(3,2,1,0) = \rho$, one-half the sum of the positive roots, and A permutes the positive roots.

(7) Consider the following game on a simply laced Dynkin diagram. Color each node black or white. If a node is black, you can toggle the colors of all *adjacent* nodes. Two colorings are said to be equivalent if you can relate them by a series of such operations.

(a) Show that in type A_n every coloring (with at least one black node) is equivalent to one with exactly 1 black node.

(b) Show that in type E_8 there are exactly three equivalence classes of colorings, one with all white nodes, and the others with one black node.

Solution: Left up to you. Search for *sigma game on non-degenerate graphs* on google.