The Theta Correspondence over $\mathbb{R}$

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1 Introduction

These notes were written for the workshop on the Theta correspondence held at the University of Maryland, May 1994. They have circulated informally for many years. Except for minor changes and some improvements to the references they have not been changed. There have been too many developments in the field since 1994 to try to bring them up to date.

This is dedicated to Roger Howe, who introduced me to representation theory in graduate school in 1977.

2 Fock model: complex Lie algebra

The Fock model is the algebraic oscillator representation: it is the $(\mathfrak{g}, \widetilde{K})$ module of the oscillator representation of $\widetilde{Sp}(2n, \mathbb{R})$. The basic references are [6],[7], and [47]. See also [11],[17] and [18]. The presentation here is due to Steve Kudla (unpublished); a similar treatment is found in [49], as well as [51]. In this section we define the oscillator representation of the complex Heisenberg and symplectic Lie algebras.

Let ψ be a non–trivial linear form on $\mathbb{C}$, i.e. fix $\lambda \in \mathbb{C}^\times$ and let $\psi(z) = \lambda z$ $(z \in \mathbb{C})$.

Definition 2.1 *Let W be a vector space with a symplectic form $\langle , \rangle$, possibly degenerate. Let*

$$\Omega(W) = T(W)/I$$

where T is the tensor algebra of W and I is the two–sided ideal generated by elements of the form

$$v \otimes w - w \otimes v - \psi\langle v, w\rangle \quad (v, w \in W).$$

This is an associative algebra, sometimes referred to as the *quantum algebra.*

For example if $\langle v, w\rangle = 0$ for all v, w, then $\Omega(W)$ is isomorphic to the polynomial algebra on W.

Let $\mathfrak{h}$ be the complex Heisenberg Lie algebra of dimension $2n+1$, so $\mathfrak{h} = W \oplus L$ where $W \simeq \mathbb{C}^{2n}$ is a vector space with a non–degenerate symplectic form $\langle , \rangle$, and $L \simeq \mathbb{C}$ is the center of $\mathfrak{h}$. Write $L = \mathbb{C}\mathbb{1}$; the Lie algebra structure of $\mathfrak{h}$ is given by $[v, w] = \langle v, w\rangle\mathbb{1}$ for $v, w \in W$. We consider ψ as an element of $Hom(L, \mathbb{C})$ by $\psi(\lambda\mathbb{1}) = \psi(\lambda)$ $(\lambda \in \mathbb{C})$. Let $\mathcal{U}(\mathfrak{h})$ be the universal enveloping algebra of $\mathfrak{h}$.

Lemma 2.2

$$\Omega(W) \simeq \mathcal{U}(\mathfrak{h})/\langle \mathbb{1} - \psi(\mathbb{1})\rangle$$

where $\langle \mathbb{1} - \psi(\mathbb{1})\rangle$ is the two–sided ideal generated by the element $\mathbb{1} - \psi(\mathbb{1})$.

Proof. Define a map $\mathfrak{h} \to \Omega(W)$ by $w \to \overline{w} \in T(W)/I$ and $\mathbb{1} \to \overline{\psi(\mathbb{1})}$. This extends to a map $\mathcal{U}(\mathfrak{h}) \to T(W)/I$ by the universal property of $\mathcal{U}(\mathfrak{h})$, and factors to the quotient; it is easily seen to be bijective. □

Note: The definition of $\Omega(W)$ depends on ψ. Over $\mathbb{C}$ the algebras defined by any ψ and ψ' are isomorphic. The isomorphism takes $v \in W$ to λv where $\lambda^2 = \psi(1)/\psi'(1)$. Over an arbitrary field F there are $|F^\times/F^{\times 2}|$ isomorphism classes of such algebras.

Since $T(W) = T^0(W) \oplus T^1(W) \oplus \ldots$ is graded, $\Omega(W)$ is filtered, i.e. $\Omega^0(W) \subset \Omega^1(W) \subset \ldots$ where $\Omega^k(W)$ consists of those elements which may be represented by some $X \in T^0(W) \oplus T^1(W) \oplus \cdots \oplus T^k(W)$. Then if $[A, B] = A \otimes B - B \otimes A$ is the commutator bracket in $\Omega(W)$, it follows easily that

$$[\Omega^k(W), \Omega^\ell(W)] \subset \Omega^{k+\ell-2}(W).$$

It follows that $\Omega^2(W)$ with the bracket $[,]$ is a Lie algebra, and $\Omega^1(W)$ is an ideal in $\Omega^2(W)$. Similarly $[\Omega^1(W), \Omega^1(W)] \subset \Omega^0(W)$, and $\Omega^1(W)$ is a two–step nilpotent Lie algebra.

We therefore have the exact sequences of Lie algebras

(2.3)(a) $$0 \to \Omega^0(W) \to \Omega^1(W) \to \Omega^1(W)/\Omega^0(W) \to 0$$

(2.3)(b) $$0 \to \Omega^1(W) \to \Omega^2(W) \to \Omega^2(W)/\Omega^1(W) \to 0$$

If $\langle , \rangle$ non–degenerate it is easy to see $\Omega^1(W) \simeq \mathfrak{h}$ and $\Omega^1(W)/\Omega^0(W) \simeq W$ (as an abelian Lie algebra). The exact sequence (2.3)(a) is isomorphic to $\mathbb{C} \to \mathfrak{h} \to W$, which splits as vector spaces (but not as Lie algebras). Something stronger holds for (2.3)(b):

Lemma 2.4 *Let $\mathfrak{g} \simeq \mathfrak{sp}(2n, \mathbb{C})$ be the Lie algebra of $Sp(W) \simeq Sp(2n, \mathbb{C})$.*

(a) $\Omega^2(W)/\Omega^1(W)$ is isomorphic to $\mathfrak{g}$.

(b) The exact sequence (1.3)(b) splits, to give an isomorphism

$$\Omega^2(W) \simeq \mathfrak{h} \oplus \mathfrak{g}$$

where the right–hand side is the semi–direct product of Lie algebras, in which $\mathfrak{g}$ acts on $\mathfrak{h}$ by

$$X \cdot (w, t) = (Xw, t)$$

for $X \in \mathfrak{g}, w \in W, t \in L$.

Define an action of $\Omega^2(W)$ on $\Omega^1(W)/\Omega^0(W)$ by $X \cdot Y\Omega^0(W) = [X, Y]\Omega^0(W)$. This is defined on $\Omega^1(W)$ since $[\Omega^2(W), \Omega^1(W)] \subset \Omega^1(W)$ and passes to the quotient since $[\Omega^2(W), \Omega^0(W)] \subset \Omega^0(W)$ (in fact this bracket is 0). By the isomorphism $\Omega^1(W)/\Omega^0(W) \simeq W$ this defines a map $\Omega^2(W) \to \mathrm{End}(W)$. If $v, w \in W$ write vw for the coset of $v \otimes w \in T(W)$, modulo(I). Explicitly for $v_1, v_2, w \in W$ we have

$$\begin{aligned} v_1 v_2 \cdot w &= v_1 v_2 w - w v_1 v_2 \\ &= v_1 v_2 w - v_1 w v_2 - \psi\langle w, v_1 \rangle v_2 \\ &= v_1 v_2 w - v_1 v_2 w - \psi\langle w, v_2 \rangle v_1 - \psi\langle w, v_1 \rangle v_2 \\ &= \psi\langle v_1, w \rangle v_2 + \psi\langle v_2, w \rangle v_1. \end{aligned}$$

It follows readily that the image of $\Omega^2(W)$ is in $\mathfrak{sp}(W)$. Furthermore this map is injective, and by dimension counting it is an isomorphism, proving (a).

Choose a standard basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of W, i.e. $\langle e_i, f_i \rangle = 1$ and all other brackets are 0. Consider the subspace of $\Omega^2(W)$ spanned by the following elements. Note that in $\Omega(W)$ all the $e_j's$ commute among themselves, similarly the $f_j's$, and $e_i f_j = f_j e_i + \psi(\delta_{i,j})$. Let

$$\text{(2.5)(a)} \qquad \begin{aligned} X^+_{i,j} &= f_i f_j \quad 1 \le i \le j \le n \\ X^-_{i,j} &= e_i e_j \quad 1 \le i \le j \le n \\ Z_{i,j} &= \frac{1}{2}(e_i f_j + f_j e_i) \quad 1 \le i, j \le n \end{aligned}$$

Note that

$$\text{(2.5)(b)} \qquad \begin{aligned} Z_{i,j} = \frac{1}{2}(f_j e_i + e_i f_j) &= \frac{1}{2}(f_j e_i + e_i f_j - f_j e_i + f_j e_i) \\ &= f_j e_i + \frac{1}{2}[e_i, f_j] \\ &= f_j e_i + \frac{1}{2}\psi(\delta_{i,j}) \end{aligned}$$

We claim the elements (2.5)(a) span a Lie subalgebra of Ω^2 isomorphic to $\mathfrak{sp}(2n, \mathbb{C}) = \mathfrak{g}$. To see this, recall (cf. [13], for example) $\mathfrak{g}$ has the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$ where $\mathfrak{k} \simeq \mathfrak{gl}(n, \mathbb{C})$ is the complexified Lie algebra of a maximal compact subgroup K of $Sp(2n, \mathbb{R})$. The $Z_{i,j}$ span an algebra isomorphic to $\mathfrak{k}$, with a Cartan subalgebra $\mathfrak{t}$ the span of the $Z_{i,i}$. Let ϵ_i denote the element of $\mathfrak{t}^*$ satisfying $\epsilon_i(Z_{j,j}) = \psi(\delta_{i,j})$. Then $Z_{j,k}$ $(j \neq k)$ is a root vector of weight $\epsilon_j - \epsilon_k$:

$$\text{(2.5)(c)} \qquad \begin{aligned} &= [f_i e_i + \frac{1}{2}\psi(1), f_j e_k] \quad \text{(by (2.5)(b))} \\ &= [f_i e_i, f_j e_k] \\ &= f_i e_i f_j e_k - f_j e_k f_i e_i \\ &= f_i f_j e_i e_k + f_i e_k \psi(\delta_{i,j}) - f_j f_i e_i e_k - f_j e_i \psi(\delta_{i,k}) \\ &= \psi(\delta_{i,j} - \delta_{i,k}) f_j e_k \\ &= (\epsilon_j - \epsilon_k)(Z_{i,i}) Z_{j,k}. \end{aligned}$$

Similarly $X^\pm_{i,j}$ is a root vector of weight $\pm(\epsilon_i + \epsilon_j)$, so the $X^\pm_{i,j}$ span $\mathfrak{p}^\pm$. It follows from similar calculations that $[\mathfrak{k}, \mathfrak{p}^\pm] = \mathfrak{p}^\pm$, $[\mathfrak{p}^+, \mathfrak{p}^-] = [\mathfrak{k}, \mathfrak{k}] = \mathfrak{k}$, so this is a Lie algebra. From the description of the roots, the root system is of type C_n, proving Lemma 2.4.

For later use we note that the $Z_{i,j}$ with $i < j$ together with the $X^+_{i,j}$ are positive root vectors of $\mathfrak{t}$ in $\mathfrak{g}$.

Now let $W = X \oplus Y$ be a complete polarization of W (this is implicit in our choice of standard basis as in the proof of Lemma 2.4). Thus X and Y are Lagrangian planes, i.e. maximal isotropic subspaces of W. Since $\langle , \rangle$ restricted to Y is trivial, $\Omega(Y) \simeq \mathbb{C}[Y]$, the polynomial algebra on Y.

By the Poincare–Birkhoff–Witt theorem [24] $\Omega(W) = \Omega(Y) \oplus \Omega(W)X$. Let $e_1, \dots, e_n$ be a basis of X, and $f_1, \dots f_n$ be a basis of Y. This says that any element of $\Omega(W)$ may be written as a sum of monomials, where each monomial is in terms of $f_i' s$ only, or has some e_j on the right. It follows immediately that

$$\Omega(W)/\Omega(W)X \simeq \Omega(Y) \simeq \mathbb{C}[Y]. \tag{2.6}$$

This carries a representation of $\Omega(W)$ by multiplication on the left. By Lemma 1.4 this defines by restriction a representation of $\mathfrak{h}$ and $\mathfrak{g}$, or equivalently $\mathcal{U}(\mathfrak{h})$ and $\mathcal{U}(\mathfrak{g})$, on $\mathbb{C}[Y]$. The center of $\mathfrak{h}$ acts by the character ψ.

Definition 2.7 *The Fock model of the oscillator representation of $\mathfrak{h} \oplus \mathfrak{g}$ is the vector space $\mathcal{F} = \mathbb{C}[Y]$, where $\Omega(W)$ acts by multiplication on the left and the isomorphism (1.6), and $\mathfrak{h} \oplus \mathfrak{g}$ acts via the embedding in $\Omega(W)$ (Lemma 1.4(b)).*

This definition depends on the choice of ψ and complete polarization $W = X \oplus Y$, and we will write $\mathcal{F}_\psi$ on occasion. The isomorphism class is independent of these choices. Over an arbitrary field $\mathbb{F}$ the number of isomorphism classes is $|\mathbb{F}^\times/\mathbb{F}^{\times 2}|$ (see the note following Lemma 1.2).

To be explicit, with the basis e_i, f_j above, let X (resp. Y) be the span of the $e_i' s$ (resp. $f_j' s$). Then $\mathcal{F} = \mathbb{C}[f_1, \dots, f_n]$ is the space of polynomials in the f_i. The action of $f_i \in \mathfrak{h}$ is given by $f_i \cdot f_j = f_i f_j$. To compute the action of e_j, in $\Omega(W) = T(W)/I$ write

$$\begin{aligned} e_j f_i &= e_j f_i - f_i e_j + f_i e_j \\ &= \psi(\langle e_j, f_i \rangle) + f_i e_j \\ &= \psi(\delta_{i,j}) + f_i e_j \end{aligned} \tag{2.8)(a}$$

Passing to $\mathcal{F}$ by the isomorphism (1.6) the term $f_i e_j$ is zero, and thus

$$e_j \cdot f_i = \psi(\delta_{i,j}). \tag{2.8)(b}$$

Thus to compute the action of the $e_i's$ multiply on the left, then move all $e_i's$ terms past all $f_j's$ to the right using the defining relation of Ω, and finally all terms with $e_i's$ on the right are 0.

Under the isomorphism with a subspace of Ω^2 the decomposition $\mathfrak{g} = \mathfrak{p}^+ \oplus \mathfrak{k} \oplus \mathfrak{p}^-$ becomes $S^2(Y) \oplus (X \otimes Y) \oplus S^2(X)$. Here $S^2(Y)$ acts by operators which raise degree by 2, $S^2(X)$ lowers degree by 2, and $X \otimes Y$ acts by degree preserving operators.

This is a version of the canonical commutation relations. To see this we identify $\mathbb{C}[Y]$ with the polynomial functions $\mathbb{C}[z_1, \dots, z_n]$ on Y via the dual variables $z_i(f_j) = \delta_{i,j}$. Write $\mathcal{F} = \sum \mathcal{F}^n$ where $\mathcal{F}^n$ is the polynomials of degree n. Then $f_i \in W \subset \mathfrak{h}$ acts by multiplication by z_i, e_j acts by $\psi(1)\frac{d}{dz_j}$, and $L \subset \mathfrak{h}$ acts by the character ψ. The image of $\mathcal{U}(\mathfrak{h})$ is the Weyl algebra of polynomial coefficient differential operators. More precisely, identify X and Y with abelian subalgebras of $\mathfrak{h}$. Then

$$\begin{aligned} X &\to \{\frac{d}{dz_i}\} \\ Y &\to \{z_i\}. \end{aligned}$$

The action of $\mathfrak{g}$ is given by:

$$\begin{aligned} \mathfrak{p}^+ \simeq S^2(Y) &\to \{z_i z_j\} \\ \mathfrak{p}^- \simeq S^2(X) &\to \{\frac{d^2}{dz_i dz_j}\} \\ \mathfrak{k} \simeq X \otimes Y &\to \{\frac{1}{2}(z_i \frac{d}{dz_j} + \frac{d}{dz_j} z_i) = z_i \frac{d}{dz_j} + \frac{1}{2}\delta_{i,j}\}. \end{aligned}$$

Lemma 2.9 *(1) $\mathcal{F}$ is an irreducible representation of $\mathfrak{h}$, with central character ψ. The space of vectors annihilated by $X \subset \mathfrak{h}$ is the one–dimensional space of constants.*

(2) As a representation of $\mathfrak{g}$, $\mathbb{C}[Y]$ is the direct sum of two irreducible representations $\mathcal{F}_\pm$ of even and odd functions. The space of vectors in $\mathcal{F}_+$ (resp. $\mathcal{F}_-$) annihilated by $S^2(X) \subset \mathfrak{g}$ is $\mathcal{F}^0$ (resp. $\mathcal{F}^1$), of dimension 1 *(resp. n).*

The proof is immediate.

Remark: Suppose V is a vector space with a non–degenerate *orthogonal* form $(,)$, and let $\Omega(V) = T(V)/\langle v \otimes w + w \otimes v - \psi(v,w)\rangle$ (cf. Definition 1.1). Then Ω is isomorphic to a quotient of the Cayley algebra [5] (compare Lemma 2.2) and this construction produces the *spin* module of $\mathfrak{so}(V)$. There exists a theory of dual pairs in $Spin(V)$ and the duality correspondence for the spin representation. See [20],[22], [48],[1].

Remark: The action of $\mathfrak{k} \simeq \mathfrak{gl}(n,\mathbb{C})$ on $\mathcal{F} \simeq \mathbb{C}[\mathbb{C}^n]$ is by the "standard" action on $\mathbb{C}^n$, twisted by a character. Up to a character this is the derived representation of the action of $GL(n,\mathbb{C})$ acting on polynomials by $g \cdot f(v) = f(g^{-1}v)$. This is the connection between the oscillator representation and classical invariant theory [55],[20].

For use in Section 4 we record a generalization of this construction. Formally the operators (1.8) can be made to act on spaces of functions other than polynomials. For example, it is immediate that for any $\gamma \in \mathbb{C}$ the space

$$\{Pe^{\gamma \sum z_i^2} \mid P \in \mathbb{C}[z_1, \dots, z_n]\}$$

is invariant under this action, and so defines a representation of $\mathfrak{h}$ and $\mathfrak{g}$.

3 Schrödinger model

See the lectures of Brooks Roberts [43], and the references there; also see [19], [17].

We now let W_0 be a real vector space of dimension $2n$ with a non–degenerate symplectic form $\langle,\rangle$. Let $Sp(W_0) \simeq Sp(2n,\mathbb{R})$ be the isometry group of this form, and $\widetilde{Sp}(2n,\mathbb{R})$ its connected two–fold cover. Let $K \simeq U(n)$ be a maximal compact subgroup of $Sp(2n,\mathbb{R})$, and $\widetilde{K}$ its inverse image in $\widetilde{Sp}(2n,\mathbb{R})$. Then $\widetilde{K}$ is connected, and is isomorphic to the $det^{\frac{1}{2}}$ cover of K. That is, it is isomorphic to the subgroup of $U(n) \times \mathbb{C}^\times$ consisting of pairs (g,z) such that $det(g) = z^2$. This has a two–to–one map $p : (g,z) \to g$ onto $U(n)$, and a genuine character $det^{\frac{1}{2}} : (g,z) \to z$ whose square is det.

Fix a non–trivial unitary additive character ψ of $\mathbb{R}$, and let $\omega = \omega_\psi$ be the corresponding oscillator representation of $\widetilde{Sp}(2n,\mathbb{R})$ on a Hilbert space. If $W_0 = X_0 \oplus Y_0$ is a complete polarization of W_0 we may realize ω on $L^2(Y_0)$. This is the direct sum of two irreducible representations.

There is a vector Γ in ω_ψ, unique up to scalars, which is an eigenvector for the action of $\widetilde{K}$. In the usual coordinates this is the Gaussian $e^{-\frac{1}{2}\sum x_i^2}$.

For $a \in \mathbb{R}$ write $a \cdot \psi(x) = \psi(ax)$. Then the oscillator representations defined by ψ and $a^2 \cdot \psi$ are isomorphic. Up to isomorphism there are two oscillator representations of $\widetilde{Sp}(2n, \mathbb{R})$.

The $\widetilde{K}$–finite vectors $\omega_{\widetilde{K}}$ in ω are a dense subspace. The action of $\widetilde{Sp}(2n, \mathbb{R})$ lifts to an action of the real Lie algebra $\mathfrak{g}_0 = \mathfrak{sp}(2n, \mathbb{R})$ on $\omega_{\widetilde{K}}$, and this extends by complex linearity to $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$. Furthermore $\widetilde{K}$ acts on this space in a way compatible with the action of $\mathfrak{g}$. The action of $\widetilde{K}$ is *locally finite*: the subspace generated by any vector under the action of $\widetilde{K}$ is finite dimensional. Finally the the multiplicity of any $\widetilde{K}$–type is finite. (See Section 9, Example 2 for a further discussion of the $\widetilde{K}$–structure of ω_ψ). This says that the $\widetilde{K}$–finite vectors in ω are a $(\mathfrak{g}, \widetilde{K})$–module [52],[27].

4 Fock model: real Lie algebra

The discussion in section 2 made no reference to the real Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$, or (equivalently) the group K. In this section we describe the $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ module arising as the $(\mathfrak{g}, \widetilde{K})$–module of the oscillator representation ω of $\widetilde{Sp}(2n, \mathbb{R})$.

Let W_0 and $Sp(W_0) \simeq Sp(2n, \mathbb{R})$ be as in Section 2. Let $\mathfrak{g}_0 = \mathfrak{sp}(W_0) \simeq \mathfrak{sp}(2n, \mathbb{R})$ be the real Lie algebra of G. Let $W = W_0 \otimes_{\mathbb{R}} \mathbb{C}$, and extend $\langle , \rangle$ to W by linearity. We then obtain $Sp(W) \simeq Sp(2n, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{sp}(W) \simeq \mathfrak{sp}(2n, \mathbb{C})$ as in Section 2. Given a complete polarization $W = X \oplus Y$ of W and $\psi : \mathbb{C} \to \mathbb{C}$ we obtain the oscillator representation of $\mathfrak{g}$ on $\mathcal{F} = \mathbb{C}[Y]$. The isomorphism class of $\mathcal{F}$ as a representation of $\mathfrak{g}$ is independent of the choice of complete polarization. However as a representation of $\mathfrak{g}_0$ this depends on the position of this polarization with respect to the real structure.

Totally complex polarization: We assume first that X is a totally complex subspace of W, i.e. $X \cap W_0 = 0$. See [7], also [36]. The choice of such X is equivalent to the choice of a compatible complex structure J on W_0: i.e. an element $J \in Sp(W_0)$ satisfying $J^2 = -1$. Given such J it extends by complex linearity to W, and we let X be the i–eigenspace of J. Conversely given X there exists unique $J \in Sp(2n, \mathbb{R})$ such that $J|_X = iId$. Note that $Y = \overline{X}$ is the $-i$–eigenspace of J, and $W = X \oplus Y$ is a complete polarization of W.

Now J makes W_0 a complex vector space of dimension n (with multiplication by i given by J). We denote this space W_J: this is a complex vector space of dimension n equal as a set to W_0. Furthermore $\{v, w\} = \langle v, Jw\rangle - i\langle v, w\rangle$ is a non–degenerate Hermitian form on W_J. We assume this form is positive definite, i.e. of signature $(n, 0)$.

To be explicit, if $e_1, \dots, f_n$ is a standard basis of W_0, we can take $X = \langle e_1+if_1, \dots, e_n+if_n\rangle$, and then $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. That is $Je_i = -f_i, Jf_i = e_i$. More generally we could take X spanned by some $e_i + if_i$ and some $e_j - if_j$. The signature of $\{,\}$ is then (p, q) where p (resp. q) is the number of plus (resp. minus) signs.

Let $K = \{g \in Sp(W_0) | gJ = Jg\}$. Equivalently $\{gv, gw\} = \{v, w\}$ for all $v, w \in W_J$, so $K \simeq U(n, 0)$, and K is a maximal compact subgroup of $Sp(W_0)$. Furthermore this is equivalent to the condition $gX = X, gY = Y$, i.e. K is the stabilizer of the complete polarization. The corresponding complexified Lie algebra $\mathfrak{k}$ is isomorphic to $\mathfrak{gl}(n, \mathbb{C})$, and its intersection $\mathfrak{k}_0$ with $\mathfrak{g}_0$ is isomorphic to $\mathfrak{u}(n)$.

As in Section 2 consider the Fock model of the oscillator representation on $\mathbb{C}[Y]$. Then $\mathfrak{k} \simeq X \otimes Y$ acts by operators which preserve degree (cf. Section 2), and therefore $\mathfrak{k}$ acts locally finitely. The action of $\mathfrak{k}_0$ exponentiates to a certain two–fold cover $\widetilde{K}$ of K (see Section 2). It follows that $\mathbb{C}[Y]$ is a $(\mathfrak{g}, \widetilde{K})$–module. We have $\mathfrak{p}^+ \simeq S^2(Y)$ acts by operators of degree 2, and $\mathfrak{p}^- \simeq S^2(X)$ by operators of degree -2.

Real polarization: At the opposite extreme we could take a complete polarization of W to be the complexification of one of W_0. To distinguish this case from the previous one we write $W_0 = \mathcal{X}_0 \oplus \mathcal{Y}_0$ and $W = \mathcal{X} \oplus \mathcal{Y}$, so $\mathcal{X} = \mathcal{X}_0 \otimes_{\mathbb{R}} \mathbb{C}$, $\mathcal{X}_0 = \mathcal{X} \cap W_0$, and similarly for $\mathcal{Y}$. We can then define the representation of $\mathfrak{g}$ on $\mathcal{F} = \mathbb{C}[\mathcal{Y}]$. The subalgebra of $\mathfrak{g}$, i.e. $\mathcal{X} \otimes \mathcal{Y}$ which acts locally finitely is again isomorphic to $\mathfrak{gl}(n, \mathbb{C})$, but the intersection with $\mathfrak{g}_0$ is then $\mathfrak{gl}(n, \mathbb{R})$. Assume $exp(\psi) : \mathbb{R} \to \mathbb{C}^\times$ is unitary, i.e. $\psi(z) = \lambda z$ for $\lambda \in i\mathbb{R}$.

This is essentially the Schrödinger model. Recall (§3) in the Schrödinger model the Gaussian $\Gamma = e^{-\frac{1}{2}\sum x_i^2}$ is an eigenvector for the action of $\widetilde{K}$, or equivalently $\mathfrak{k}$.

In the Fock model on $\mathbb{C}[Y]$, the constants are the unique eigenspace for $\mathfrak{k} \simeq X \otimes Y$. The constants are also characterized as the kernel of the action of $X \subset \mathfrak{h}$. Therefore we look for a vector in the Schrödinger model on $\mathbb{C}[\mathcal{Y}]$ annihilated by X. Choose a basis of $\mathcal{Y}$ with dual coordinates $z_1, \dots, z_n$ so that $\mathbb{C}[Y] \simeq \mathbb{C}[z_1, \dots, z_n]$. Recall (1.10) we can extend the action on $\mathbb{C}[z_1, \dots, z_n]$ to polynomials times a Gaussian.

Lemma 4.1

1. *There is a unique value of γ for which the Gaussian*
$$\Gamma = e^{-\gamma\frac{1}{2}\sum z_i^2}$$
is annihilated by $X \subset \mathfrak{h}$.

2. *The space*
$$\mathbb{C}[\mathcal{Y}]\Gamma = \{Pe^{-\frac{1}{2}\gamma\sum z_i^2} \mid P \in \mathbb{C}[z_1, \dots, z_n]\}$$
is a $(\mathfrak{g}, \widetilde{K})$–module,

3. *$\mathbb{C}[Y]$ and $\mathbb{C}[\mathcal{Y}]\Gamma$ are isomorphic as $\mathfrak{h}$ and as $(\mathfrak{g}, \widetilde{K})$–modules.*

4. *Given $\mathcal{Y}$ we can choose Y (totally complex) so that*
$$\Gamma = e^{-\frac{1}{2}\sum z_i^2}.$$
Then $\mathbb{C}[Y]$ is isomorphic to the space of $\widetilde{K}$–finite vectors $L^2(\mathcal{Y}_0)_{\widetilde{K}}$ in the Schrödinger model $L^2(\mathcal{Y}_0)$ of the oscillator representation.

Proof. This essentially reduces to $SL(2)$. Let e, f be a standard basis of W_0, and let $\mathcal{X} = \mathbb{C}\langle e\rangle, \mathcal{Y} = \mathbb{C}\langle f\rangle$. Let $X = \mathbb{C}\langle e + i\nu f\rangle, Y = \mathbb{C}\langle e - i\nu f\rangle$ for some $0 \neq \nu \in \mathbb{C} - i\mathbb{R}$. Let $\lambda = \psi(1)$. With dual coordinate z on $\mathcal{Y}$ satisfying $z(gf) = 1$ we have $e \in \mathcal{X}$ acts on $\mathbb{C}[\mathcal{Y}]$ by $\lambda\frac{d}{dz}$, and $f \in \mathcal{Y}$ acts by multiplication by z. Therefore $e + \nu i f$ acts by $\tau = i\nu z + \lambda\frac{d}{dz}$, and $\tau \cdot e^{\gamma z^2} = (2\gamma\lambda + i\nu)ze^{\gamma z^2}$. Thus taking
$$\gamma = \frac{-i\nu}{2\lambda}$$
gives $\tau \cdot e^{\gamma z^2} = 0$. It is immediate that γ is determined uniquely. This proves (1), and (2) follows readily from this.

It is an easy argument by induction on degree that $\mathbb{C}[\mathcal{Y}]\Gamma$ is irreducible as a representation of $\mathfrak{h}$. Since the same holds for $\mathbb{C}[Y]$, and the central characters agree, by the Stone–von Neumann theorem these are isomorphic as representations of $\mathfrak{h}$, and (3) follows immediately from this.

For (4) choose $\nu = -i\lambda$, so $\gamma = -\frac{1}{2}$ and Γ has the stated form. Finally restricting the functions in $\mathbb{C}[\mathcal{Y}]\Gamma$ to $\mathcal{Y}_0$ gives functions in $L^2(\mathcal{Y}_0)$, and this gives the isomorphism of (4). $\square$

Remark: Suppose $\mathcal{X} = \langle e_1, \dots, e_n\rangle$, $\mathcal{Y} = \langle f_1, \dots, f_n\rangle$ as usual, and $X = \langle e_1 + if_1, \dots, e_n + if_n\rangle$, $Y = \langle e_1 - if_1, \dots, e_n - if_n\rangle$. Also assume $\psi(1) = i$ and $\nu = 1$. Then K is the usual maximal compact subgroup of $Sp(2n, \mathbb{R})$, and this is how the Fock/Schrödinger models are usually stated.

Remark: As far as the actions of the Lie algebras are concerned, we see that both the Fock and Schrödinger models are examples of the construction of Section 2, extended to polynomials times the Gaussian in the latter case. However the full Hilbert spaces in the two cases look somewhat different. In the Fock model this consists of holomorphic functions on Y which are square integrable with respect to a certain measure [7], whereas in the Schrödinger model this is $L^2(\mathcal{Y}_0)$. The existence of these two isomorphic models accounts for the simultaneously analytic and algebraic nature of the oscillator representations.

This duality is nicely illustrated by looking at the operators in the two models in the case of $SL(2)$. We use the notation of the proof of Lemma 4.1, with $\lambda = i$ and $\nu = 1$. We write $\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$, or alternatively as $= \mathfrak{m} \oplus \mathfrak{n} \oplus \overline{\mathfrak{n}}$.

Fock Model:

Here the space of $\widetilde{K}$–finite vectors in the oscillator representation is $\mathcal{F} = \mathbb{C}[Y] = \mathbb{C}[z]$.

$\mathfrak{k} \oplus \mathfrak{p}$ acts by:

$$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \in \mathfrak{k} \to -z\frac{d}{dz} - \frac{1}{2}$$

$$\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \in \mathfrak{p}^+ \to z^2$$

$$\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \in \mathfrak{p}^- \to -\frac{d^2}{dz^2}$$

And the operators from $\mathfrak{m} \oplus \mathfrak{n} \oplus \overline{\mathfrak{n}}$:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{m} \to \frac{1}{2}(z^2 - \frac{d^2}{dz^2})$$
$$\begin{pmatrix} 0 & 4i \\ 0 & 0 \end{pmatrix} \in \mathfrak{n} \to -2z\frac{d}{dz} - 1 + z^2 + \frac{d^2}{dz^2}$$
$$\begin{pmatrix} 0 & \\ -4i & 0 \end{pmatrix} \in \overline{\mathfrak{n}} \to -2z\frac{d}{dz} - 1 - z^2 - \frac{d^2}{dz^2}$$

Schrödinger Model:

Now the $\widetilde{K}$–finite vectors are polynomials in x times the Gaussian $e^{-\frac{1}{2}x^2}$. Here the formulas for $\mathfrak{m} \oplus \mathfrak{n} \oplus \overline{\mathfrak{n}}$ are easy:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{m} \to x\frac{d}{dx} + \frac{1}{2}$$
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{n} \to \frac{i}{2}x^2$$
$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \overline{\mathfrak{n}} \to \frac{i}{2}\frac{d}{dx^2}$$

whereas $\mathfrak{k} \oplus \mathfrak{p}$ acts by:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{k} \to \frac{i}{2}(x^2 - \frac{d^2}{dx^2})$$
$$\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \in \mathfrak{p}^+ \to (x\frac{d}{dx} + \frac{1}{2}) - \frac{1}{2}x^2 - \frac{1}{2}\frac{d^2}{dx^2}$$
$$\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \in \mathfrak{p}^- \to (x\frac{d}{dx} + \frac{1}{2}) + \frac{1}{2}x^2 + \frac{1}{2}\frac{d^2}{dx^2}$$

We look at the isomorphism between the Fock and Schrödinger models more closely. It takes the constants in the Fock model to the multiples of the Gaussian in the Schrödinger model. It intertwines the actions of Y, for which these vectors are cyclic, and the isomorphism is determined by these conditions.

Consider the example of $SL(2)$ in the proof of Lemma 4.1, and the action of $e - \nu i f \in Y$ on $\mathbb{C}[Y]$ and $\mathbb{C}[\mathcal{Y}]\Gamma$; the isomorphism intertwines these actions. With $\nu = -i\lambda$ let z dual to $e - \lambda f$ be the coordinate on Y and x dual to f

on $\mathcal{Y}$. We conclude multiplication by z goes to $\lambda(\frac{d}{dx}-x)$ on $\mathbb{C}[\mathcal{Y}]\Gamma$. Thus the isomorphism takes

$$z^n \to \lambda^n(\frac{d}{dx}-x)^n \cdot e^{-\frac{1}{2}x^2}.$$

Note that $(\frac{d}{dx}-x)(x^n\Gamma) = (-2x^{n+1} + \text{ lower order terms})\Gamma$. It follows that this isomorphism takes

$$z^n \to p_n(x)e^{-\frac{1}{2}x^2}$$

where $p_n(z) = ((-2\lambda z)^n + \text{lower order terms})e^{-\frac{1}{2}z^2}$. The following Lemma follows readily from this discussion.

Lemma 4.2 *Given ψ and $W_0 = \mathcal{X}_0 \oplus \mathcal{Y}_0$, let $L^2(\mathcal{Y}_0)$ be the corresponding oscillator representation, with $\widetilde{K}$–finite vectors $L^2(\mathcal{Y}_0)_{\widetilde{K}}$. Let $W = X \oplus Y$ be as in Lemma 3.1(4) so that there is an isomorphism of $\mathfrak{h}$ and $(\mathfrak{g}, \widetilde{K})$–modules*

$$\phi : \mathbb{C}[Y] \to L^2(\mathcal{Y}_0)_{\widetilde{K}}$$

Then we can choose coordinates $z_1, \ldots, z_n$ on Y and $x_1, \ldots, x_n$ on $\mathcal{Y}_0$ such that if $P(z_1, \ldots, z_n)$ is a polynomial of degree d then

$$\phi(P) = (c_d P(x_1, \ldots, x_n) + Q)\Gamma$$

where Q is a polynomial of degree strictly less than d, and c_d is a constant depending only on d.

In fact $\phi(z_1^{a_1} \ldots z_n^{a_n}) = (cx_1^{a_1} \ldots x_n^{a_n} + Q)\Gamma$ and up to a constant $cx_1^{a_1} \ldots x_n^{a_n} + Q$ is a *Hermite polynomial* [23].

5 Duality

The results in this section are due to R. Howe [21].

We begin by listing the irreducible reductive dual pairs over $\mathbb{R}$. This follows the general scheme of [15], and is spelled out in this case in [21]. See the lectures of Brooks Roberts [43] for more details.

The irreducible pairs of type I, together with the corresponding division algebra and involution are:

1. $(O(p,q), Sp(2n,\mathbb{R})) \subset Sp(2n(p+q),\mathbb{R})$, $\quad(\mathbb{R},1)$

2. $(U(p,q), U(r,s)) \subset Sp(2(p+q)(r+s),\mathbb{R}) \quad (\mathbb{C},^-)$

3. $(Sp(p,q), O^*(2n)) \subset Sp(4n(p+q),\mathbb{R}) \quad (\mathbb{H},^-)$

4. $(O(m,\mathbb{C}), Sp(2n,\mathbb{C})) \subset Sp(4mn,\mathbb{R}) \quad (\mathbb{C},1)$

The type II dual pairs are

1. $(GL(m,\mathbb{R}), GL(n,\mathbb{R})) \subset Sp(2mn,\mathbb{R})$

2. $(GL(m,\mathbb{C}), GL(n,\mathbb{C})) \subset Sp(4mn,\mathbb{R})$

3. $(GL(m,\mathbb{H}), GL(n,\mathbb{H})) \subset Sp(8mn,\mathbb{R})$

Notation is as in [13]. The group $GL(m,\mathbb{H})$ is sometimes called $U^*(2m)$ and is a real form of $GL(2m,\mathbb{C})$.

Now let (G,G') be a reductive dual pair in $Sp(2n,\mathbb{R})$. Fix a non–trivial unitary additive character ψ of $\mathbb{R}$, and let ω_ψ be the corresponding oscillator representation. Fix a maximal compact subgroup $\mathbb{K} \simeq U(n)$ of $Sp(2n,\mathbb{R})$ and let $\mathcal{F}_\psi \simeq \mathbb{C}[z_1,\dots,z_n]$ be the corresponding Fock model.

We may assume $K = G\cap\mathbb{K}$ and $K' = G'\cap\mathbb{K}$ are maximal compact subgroups of G and G' respectively. Then $\mathcal{F}$ is a $(\mathfrak{g}\oplus\mathfrak{g}', \widetilde{K}\cdot\widetilde{K}')$ module.

Remark 5.1 For H a subgroup of $Sp(W)$, $\widetilde{H}$ will always denote the inverse image of H in $\widetilde{Sp}(W)$, unless otherwise stated. For (G,G') a dual pair $\widetilde{G}$ commutes with $\widetilde{G}'$, and so $\widetilde{G\cdot G'} = \widetilde{G}\cdot\widetilde{G}'$.

We denote an *outer* tensor product of representations π_i of G_i as $\pi_1\check{\otimes}\pi_2$. This is a representation of $G_1\times G_2$. In the case $G=G_1=G_2$ the restriction to the diagonal subgroup G is the ordinary tensor product.

We will often abuse notation and write $\pi\check{\otimes}\pi'$ for a representation of $\widetilde{G}\cdot\widetilde{G}'$. Here π (resp. π') is a genuine representation of $\widetilde{G}$ (resp. $\widetilde{G}'$) and $\pi\check{\otimes}\pi'$ is a representation of $\widetilde{G}\times\widetilde{G}'$. There is a surjective map $\widetilde{G}\times\widetilde{G}' \to \widetilde{G}\cdot\widetilde{G}'$, and $\pi\check{\otimes}\pi'$ is trivial on the kernel, so it factors to a genuine representation of $\widetilde{G}\cdot\widetilde{G}'$. If $G\cap G'=1$ then this map is two–to–one.

Definition 5.2 *$\mathcal{R}(\mathfrak{g},\widetilde{K},\psi)$ is the set of isomorphism classes of irreducible $(\mathfrak{g},\widetilde{K})$ modules π such that there exists a non–zero $(\mathfrak{g},\widetilde{K})$–module map (necessarily a surjection) $\mathcal{F}\to\pi$.*

Theorem 5.3 ([21], Theorem 2.1) *If $\pi \in \mathcal{R}(\mathfrak{g}, \widetilde{K}, \psi)$, there exists a unique $\pi' \in \mathcal{R}(\mathfrak{g}', \widetilde{K}', \psi)$ such that*

$$Hom_{\mathfrak{g}\oplus\mathfrak{g}', \widetilde{K}\cdot\widetilde{K}'}(\mathcal{F}_\psi, \pi\check{\otimes}\pi') \neq 0.$$

In fact this Hom space is one–dimensional.

Definition 5.4 *We define a correspondence*

$$\mathcal{R}(\mathfrak{g}, \widetilde{K}, \psi) \leftrightarrow \mathcal{R}(\mathfrak{g}', \widetilde{K}', \psi)$$

by the condition $\pi \leftrightarrow \pi'$ if

$$Hom_{\mathfrak{g}\oplus\mathfrak{g}', \widetilde{K}\cdot\widetilde{K}'}(\mathcal{F}_\psi, \pi\check{\otimes}\pi') \neq 0.$$

By Theorem 5.3 this is a bijection. We refer to this as the duality correspondence or (local) theta–correspondence, and we write $\pi' = \theta(\psi)(\pi)$ and $\pi = \theta(\psi)(\pi')$.

The duality correspondence is a bijection between certain $(\mathfrak{g}, \widetilde{K})$ and $(\mathfrak{g}', \widetilde{K}')$ modules. By the standard theory of $(\mathfrak{g}, K)$–modules a module π in $\mathcal{R}(\mathfrak{g}, \widetilde{K}, \psi)$ corresponds to a genuine representation of $\widetilde{G}$. Let $\widetilde{G}\widehat{\ }_{genuine}$ denote the irreducible admissible genuine representations of $\widetilde{G}$. Then the duality correspondence defines a bijection between subsets of $\widetilde{G}\widehat{\ }_{genuine}$ and $\widetilde{G}'\widehat{\ }_{genuine}$.

Given $\pi \in \mathcal{R}(\mathfrak{g}, \widetilde{K}, \psi)$ let $\mathcal{N}(\pi)$ be the intersection of the kernels of all $\phi \in \mathrm{Hom}_{\mathfrak{g},\widetilde{K}}(\mathcal{F}_\psi, \pi)$. Then $\mathcal{N}(\pi)$ is stable under the action of $(\mathfrak{g}\oplus\mathfrak{g}', \widetilde{K}\cdot\widetilde{K}')$, and $\mathcal{F}_\psi/\mathcal{N}(\pi)$ is isomorphic to $\pi\check{\otimes}\pi_1'$ for some $(\mathfrak{g}', \widetilde{K}')$ module π_1'. The preceding Theorem follows immediately from

Theorem 5.5 ([21] Theorem 2.1) *The module π_1' is of finite length, and has a unique irreducible quotient.*

Then π' is the unique irreducible quotient of π_1'.

Theorems 5.3 and 5.5 have smooth and unitary analogues, i.e. with $\mathcal{F}$ replaced by the unitary representation of $\widetilde{Sp}(2n, \mathbb{R})$ on a Hilbert space $\mathcal{H}$, or on the smooth vectors $\mathcal{H}^\infty$ of $\mathcal{H}$. See [21].

Remark 5.6 We say a two–fold covering $p : \widetilde{G} \to G$ *splits* if the exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{G} \to G \to 1$$

splits. All of our extensions will be central, i.e. the kernel of the covering map is contained in the center of $\widetilde{G}$, so having a splitting is equivalent to $\widetilde{G} \simeq G \times \mathbb{Z}/2\mathbb{Z}$. The choice of splitting is equivalent to the choice of this isomorphism. Given such a choice there is a bijection between the irreducible genuine representations of $\widetilde{G}$ and the irreducible representations of G. This bijection is realized by tensoring with the distinguished character $\zeta = \mathbb{1}\check{\otimes} sgn$ of $G \times \mathbb{Z}/2\mathbb{Z}$.

More generally even if (4.6) is not split, there may exist a genuine character ζ of $\widetilde{G}$. For example this is the case if $\widetilde{G}$ is the $det^{\frac{1}{2}}$ cover of $GL(n)$ or a subgroup such as $O(n)$ or $U(n)$. In fact this is the case for all the groups under consideration except the metaplectic group $\widetilde{Sp}(2n, \mathbb{R})$. Tensoring with ζ then determines a bijection as in the split case.

Thus, given some choices, in the duality correspondence between $\widetilde{G}^\frown_{genuine}$ and $\widetilde{G}'^\frown_{genuine}$ in some cases we can replace $\widetilde{G}^\frown_{genuine}$ by $G^\frown$, and similarly $\widetilde{G}'$.

Open problem: *Explicitly* compute the duality correspondence of Definition 4.4.

Here "explicitly" has several possible meanings. In some sense the best solution is to compute the bijection in terms of some parameterization of representations, such as the Langlands classification or variations of it. However other descriptions are possible.

Question: How does the duality correspondence behave with respect to unitarity?

In the stable range $\theta(\psi)$ preserves unitarity going from the smaller to the larger group: if π is unitary then so is $\theta(\psi)(\pi)$ [30]. Very little is known in general. It takes Hermitian representations to Hermitian representations [37].

6 Compact dual pairs

In this section we consider the duality correspondence when one member G or G' of the dual pair (G, G') is compact. We refer to such a dual pair as a *compact* dual pair. For the general dual pair (G, G') the maximal compact subgroup K is a member of a compact dual pair (K, M'), and the duality correspondence for (K, M') plays a fundamental role in determining that for (G, G').

The duality correspondence for compact dual pairs is substantially simpler than the general case. The representations of the non–compact member of the dual pair are all (unitary) highest weight modules, and hence determined by their highest weight. (In fact all unitary highest weight modules of the classical groups arise this way [9], with a small number of exceptions in the case of $O^*(2n)$ [8]). An explicit computation of most cases is found in [25], and a closely related discussion is in [20]. In fact a number of such cases, treated in an *ad hoc* manner, provided some of the initial evidence for Howe's formalism and duality conjecture [10], [12], [44].

Let (K, G') be a dual pair in $Sp(2n, \mathbb{R})$ with K compact. Let $\mathbb{K}, \psi$ and $\mathcal{F}$ be as in Section 4. We assume $K, K' \subset \mathbb{K}$ where K' is a maximal compact subgroup of G. Recall (§2) the complexified Lie algebra of $Sp(2n, \mathbb{R})$ has the decomposition $\mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$.

The complexified Lie algebra $\mathfrak{g}'$ of G' has the decomposition $\mathfrak{g}' = \mathfrak{p}'^+ \oplus \mathfrak{k}' \oplus \mathfrak{p}'^-$ given by intersection with the corresponding decomposition of $\mathfrak{sp}(2n, \mathbb{C})$.

Theorem 6.1 *Let*

$$\mathcal{H}(K) = \{P \in \mathcal{F} \mid X \cdot P = 0 \quad \forall X \in \mathfrak{p}'^-\},$$

the space of K–harmonics.

Since the action of $\mathfrak{p}'^-$ commutes with the action of $\widetilde{K}$, $\mathcal{H}(K)$ is clearly $\widetilde{K}$–invariant.

Recall that $\widetilde{K}$ preserves the degree of polynomials in $\mathcal{F}$, so the polynomials $\mathcal{F}^n$ of degree n are a $\widetilde{K}$–stable subspace.

Definition 6.2 *Let σ be an irreducible representation of $\widetilde{K}$. The degree $d(\sigma)$ of σ is defined to be the minimal degree d such that the σ isotypic component $\mathcal{F}^d_\sigma$ of $\mathcal{F}^d$ is non–zero, or ∞ if $\mathcal{F}_\sigma$ is empty.*

If $d(\sigma) \neq \infty$, consider $\mathcal{F}_\sigma$, which is invariant by $\widetilde{K}$ and $\mathfrak{g}'$. Part of the proof of Theorem 5.5 is contained in the next result. See [20] and [21, Section 3].

Theorem 6.3

1. *$\mathcal{F}_\sigma$ is irreducible as a representation of $\widetilde{K} \times \mathfrak{g}'$. In particular $\mathcal{F}_\sigma \simeq \sigma \otimes \pi'$ for some irreducible $(\mathfrak{g}', \widetilde{K}')$–module π'.*

2. *The correspondence $\sigma \leftrightarrow \pi'$ is a bijection.*

3. *The polynomials of lowest degree (=$d(\sigma)$) in $\mathcal{F}_\sigma$ are precisely $\mathcal{H}(K)_\sigma$.*

4. *$\mathcal{F}_\sigma = \mathcal{U}(\mathfrak{p}'^{+}) \cdot \mathcal{H}(K)_\sigma$*

5. *$\mathcal{H}(K)_\sigma$ is irreducible as a representation of $\widetilde{K} \times \mathfrak{k}'$, so $\mathcal{H}(K)_\sigma \simeq \sigma \otimes \tau'$ for some irreducible representation τ' of $\mathfrak{k}'$.*

6. *The correspondence $\sigma \leftrightarrow \tau'$ is a bijection.*

In other words
$$\mathcal{F} \simeq \sum \sigma \otimes \pi'$$
as a representation of $\widetilde{K} \times \mathfrak{g}'$, and $\sigma \leftrightarrow \pi'$ is a bijection. This is a special case of Theorem 5.3. Furthermore each π' contains a unique $\widetilde{K}'$–type τ' of lowest degree, which generates π' under the action of $\mathcal{U}(\mathfrak{p}'^{+})$, and finally the correspondence $\sigma \leftrightarrow \tau'$ is also a bijection.

Thus π' is a highest weight module, in that it has a cyclic vector under the action of the Borel subalgebra $\mathfrak{b} = \mathfrak{b}_{\mathfrak{k}'} \oplus \mathfrak{p}'^{+}$, where $\mathfrak{b}_{\mathfrak{k}'}$ is a Borel subalgebra of $\mathfrak{k}'$ [24]. Such a module is completely determined by its highest weight, which is the highest weight of τ' as a representation of $\mathfrak{k}'$.

Note that $\widetilde{K}$ may be a non–trivial covering group of K, and $\widetilde{K}$ as well as K may be disconnected. However except for these technicalities, the correspondence $\sigma \leftrightarrow \pi'$ is explicitly described by the correspondence of highest weights of σ for $\widetilde{K}$ and τ' for $\mathfrak{k}'$. We proceed to do this for all irreducible compact dual pairs.

The irreducible compact dual pairs are the following.

1. $(O(p), Sp(2n,\mathbb{R})) \subset Sp(2pn,\mathbb{R})$,

2. $(U(p), U(m,n)) \subset Sp(2p(m+n),\mathbb{R})$,

3. $(Sp(p), O^*(2n)) \subset Sp(4pn,\mathbb{R})$

(An exceptional case is $(Sp(p,q), O^*(2))$ with $pq \neq 0$. Here $O^*(2) \simeq U(1)$ is compact, but Theorem 5.3 does not apply, because the decomposition preceding Definition 5.1 does not hold.)

I. $(O(p), Sp(2n,\mathbb{R}))$.

The irreducible representations of $SO(p)$ are parameterized by highest weights $(a_1, \dots, a_{[\frac{p}{2}]})$ with $a_i \in \mathbb{Z}$ and $a_1 \geq a_2 \cdots \geq a_{[\frac{p}{2}]-1} \geq |a_{[\frac{p}{2}]}|$. If p is odd we may assume $a_{[\frac{p}{2}]} \geq 0$.

Following Weyl [55] we parameterize representations of $O(p)$ by restriction from $U(p)$. Consider the irreducible representations of $U(p)$ given by Young's diagrams with p rows of length $a_1, \dots, a_p$, or equivalently with highest weight $(a_1, \dots, a_p)$ with $a_1 \geq a_2 \geq \cdots \geq a_p \geq 0$. We embed $O(p)$ in $U(p)$ in the usual way, i.e $O(p) = U(p) \cap GL(p,\mathbb{R})$.

Lemma 6.4 ([55]) *The irreducible representations of $O(p)$ are parametrized by Young diagrams Y with rows of length $a_1 \geq a_2 \geq \dots a_p \geq 0$ such that the sum of the lengths of the first two columns is less than or equal to p. The representation of $O(p)$ defined by Y is defined to be the irreducible summand of the representation of $U(p)$ defined by Y which contains the highest weight vector.*

Equivalently the highest weights of the representations of $U(p)$ are of the form

$$(a_1, \dots, a_k, 0, \dots, 0) \quad a_1 \geq \cdots \geq a_k > 0, k \leq [\frac{p}{2}]$$

or

$$(a_1, \dots, a_k, \overbrace{1, \dots, 1}^{\ell}, 0, \dots, 0), \quad a_1 \geq \cdots \geq a_k \geq 1, 2k + \ell = p.$$

In (a) the length of the first column is less than or equal to $[\frac{p}{2}]$, and we say this representation has "highest weight" $(a_1, \dots, a_k, 0, \dots, 0; 1)$. In (b) the length of the first column is greater than $[\frac{p}{2}]$ and we denote the "highest weight" of this representation $(a_1, \dots, a_k, 0, \dots, 0; -1)$.

For example the *sgn* representation of $O(p)$ is the restriction of the determinant representation of $U(p)$, which has highest weight $(1,\ldots,1)$; the corresponding "highest weight" of $O(p)$ is $(0,\ldots,0;-1)$.

The representations given by $(a_1,\ldots,a_k,0,\ldots,\epsilon)$ with $\epsilon=\pm1$ differ by tensoring with *sgn*. These two representations are not isomorphic to one another unless p is even and $a_{\frac{p}{2}}\neq 0$.

The genuine irreducible representations of the $det^{\frac{1}{2}}$ cover of $U(n)$ are parametrized by highest weights $(a_1,\ldots,a_n)$ with $a_i\in\mathbb{Z}+\frac{1}{2}$ and $a_1\geq a_2\cdots\geq a_n$.

Now the inverse image $\widetilde{K}$ of $K=O(p)$ in the metaplectic group is isomorphic to the cover defined by $det^{\frac{n}{2}}$, and is therefore trivial over the connected component $SO(p)$, and also over $O(p)$ if and only if n is even. In any event $\widetilde{O}(p)$ has a genuine character ζ, and we identify genuine representations of $\widetilde{O}(p)$ with representations of $O(p)$ by tensoring with ζ. There are two choices of ζ if n is odd. The covering $\widetilde{Sp}(2n,\mathbb{R})$ of $Sp(2n,\mathbb{R})$ is split if and only if p is even, otherwise it is the metaplectic group. The maximal compact subgroup K' of $Sp(2n,\mathbb{R})$ is isomorphic to $U(n)$. There is a choice of this isomorphism, which will be specified below. Then $\widetilde{K}'$ is the split cover of $U(n)$ if p is even, or is isomorphic to the $det^{\frac{1}{2}}$ cover $\widetilde{U}(n)$ if n is odd. Since this is connected it is equivalent to work on the Lie algebra $\mathfrak{k}'$.

The duality correspondence in this case is a correspondence between genuine irreducible representations of $\widetilde{K}$ and $\widetilde{K}'$, or equivalently $\mathfrak{k}'$. We identify genuine representations of $\widetilde{K}$ with representations of $O(p)$, and those of $\widetilde{K}'$ with $\widetilde{U}(n)$ or $\mathfrak{gl}(n,\mathbb{C})$. The choices involved are determined by the following convention.

Normalization: We choose an isomorphism of K' with $U(n)$ so that the constants $\mathcal{F}^0$ of $\mathcal{F}$ are considered the irreducible representation of $\widetilde{U}(n)$ or $\mathfrak{gl}(n,\mathbb{C})$ with highest weight $(\frac{p}{2},\ldots,\frac{p}{2})$.

Let ζ be the character by which $\widetilde{K}$ acts on $\mathcal{F}^0$. The correspondence $\sigma\rightarrow\sigma\otimes\zeta$ defines a bijection between the irreducible representations of $O(p)$ and the genuine irreducible representations of $\widetilde{K}$.

We thus describe the duality correspondence $\sigma\leftrightarrow\tau'$ in terms of representations of $O(p)$ and $\widetilde{U}(n)$ or $\mathfrak{gl}(n,\mathbb{C})$.

This normalization depends on ψ. For example it says that the highest weights of the oscillator representation defined by ψ are $(\frac{1}{2},\ldots,\frac{1}{2})$ and $(\frac{3}{2},\frac{1}{2},\ldots,\frac{1}{2})$.

Proposition 6.5 *The duality correspondence for the dual pair $(O(p), Sp(2n,\mathbb{R}))$ is given by the correspondence of highest weights $\sigma \leftrightarrow \tau'$:*

$$O(p): \quad \sigma = (a_1, \ldots, a_k, 0, \ldots, 0; \epsilon)$$

$$Sp(2n,\mathbb{R}): \quad \tau' = (a_1 + \frac{p}{2}, \ldots, a_k + \frac{p}{2}, \overbrace{\frac{p}{2}+1, \ldots, \frac{p}{2}+1}^{\frac{1-\epsilon}{2}(p-2k)}, \frac{p}{2}, \ldots, \frac{p}{2})$$

All such highest weights occur, subject to the constraints $k \leq [\frac{p}{2}]$ and $k + \frac{1-\epsilon}{2}(p-2k) \leq n$.

This means that the weight σ for $O(p)$ is the highest weight of the irreducible representation σ, and the weight for $Sp(2n,\mathbb{R})$ is the highest weight of the $\widetilde{K}'$–type of τ' of lowest degree in π'.

II. $(U(p), U(m,n))$

The inverse image $\widetilde{K}$ of $U(p)$ in $\widetilde{Sp}(2p(m+n),\mathbb{R})$ is isomorphic to the cover defined by the character $det^{\frac{m+n}{2}}$. The cover of $U(m,n)$ is the $det^{\frac{p}{2}}$ cover. Therefore the correspondence $\sigma \leftrightarrow \tau'$ is described in terms of highest weights for unitary groups or their two–fold covers, given by integers or half–integers respectively.

We assume the isomorphisms of the covers of $U(p)$ and $U(m,n)$ with the det^* covers have been chosen so the constants $\mathcal{F}^0$ have weight $(\frac{m-n}{2}, \ldots, \frac{m-n}{2})$ for $\widetilde{K}$, and weight $(\frac{p}{2}, \ldots, \frac{p}{2}) \otimes (-\frac{p}{2}, \ldots, -\frac{p}{2})$ for $\widetilde{K}'$.

Proposition 6.6 *The duality correspondence for $(U(p), U(m,n))$ is given by*

$$\begin{aligned} U(p): \quad & \sigma = (a_1, \ldots, a_k, 0, \ldots, 0, b_1, \ldots, b_\ell) + (\frac{m-n}{2}, \ldots, \frac{m-n}{2}) \to \\ U(m,n): \quad & \tau' = (a_1, \ldots, a_k, 0, \ldots, 0) \otimes (0, \ldots, 0, b_1, \ldots, b_\ell) \\ & \qquad + (\frac{p}{2}, \ldots, \frac{p}{2}) \otimes (-\frac{p}{2}, \ldots, -\frac{p}{2}). \end{aligned}$$

All such weights occur, subject to the obvious constraints $k + \ell \leq p, k \leq m, \ell \leq n$.

III. [32] $(Sp(p), O^*(2n))$.

Both $Sp(p)$ and $O^*(2n)$ are connected, and the coverings split, so we consider only the linear groups. The maximal compact subgroup of $O^*(2n)$

is isomorphic to $U(n)$.

$$\begin{aligned} Sp(p): &\quad \sigma = (a_1, \ldots, a_k, 0, \ldots, 0) \to \\ O^*(2n)): &\quad \tau' = (a_1 + p, \ldots, a_k + p, p, \ldots, p) \end{aligned}$$

All such weights occur, subject to the obvious constraints $k \leq p, n$.

7 Joint harmonics

The proof of the results in Section 5 depends heavily on information about K–types. This information is also important for computing the correspondence. The results in this section are all from [21].

We continue with the setup of Section 5. The maximal compact subgroup K of G is itself a member of a dual pair (K, M') with $K \subset G$ and $M' \supset G'$ (this is an example of a see–saw dual pair [28]) and we may apply the machinery of Section 5. The same applies *mutatis mutandis* to (M, K').

Definition 7.1 *The joint harmonics $\mathcal{H}$ is the space of polynomials which are both K–harmonic and K'–harmonic. That is*

$$\mathcal{H} = \mathcal{H}(K) \cap \mathcal{H}(K').$$

This is a $\widetilde{K} \cdot \widetilde{K}'$ invariant subspace of $\mathcal{F}$.

Let $\mathcal{R}(K, \mathcal{H})$ be the isomorphism classes of irreducible $\widetilde{K}$–modules σ such that $\mathcal{H}_\sigma \neq 0$, and define $\mathcal{R}(K', \mathcal{H})$ similarly. As in the definition of the duality correspondence, we say $\sigma \in \mathcal{R}(K, \mathcal{H})$ corresponds to $\sigma' \in \mathcal{R}(K', \mathcal{H})$ if $\sigma \check{\otimes} \sigma'$ is a direct summand of $\mathcal{H}$.

Definition 7.2 *Suppose $\pi \in \mathcal{R}(\mathfrak{g}, K, \psi)$ and σ is a $\widetilde{K}$–type occurring in π, i.e. $\pi_\sigma \neq 0$. Then we say σ is of minimal degree in π if $d(\sigma)$ is minimal among the degrees of all $\widetilde{K}$–types of π.*

Since $d(\sigma) \geq 0$, it is immediate that the set of $\widetilde{K}$–types of π of minimal degree is non–empty. The notion of $\widetilde{K}$–type of π of minimal degree depends on the dual pair, and so is not an intrinsic notion to $\mathfrak{g}$. The relationship between the $\widetilde{K}$–types of minimal degree and those which are minimal in the sense of Vogan [52] is a subtle and important one.

The following structural result is a key to the proof of Theorem 5.3, as well as to an explicit understanding of the duality correspondence.

Theorem 7.3 ([21], Section 3)

1. *The correspondence $\sigma \to \sigma'$ defined on $\mathcal{H}$ is a bijection*

$$\mathcal{R}(K, \mathcal{H}) \leftrightarrow \mathcal{R}(K', \mathcal{H}).$$

   We write $\sigma' = \theta(\psi, \mathcal{H})(\sigma)$.

2. *$\mathcal{H}$ generates $\mathcal{F}$ as a representation of $\mathfrak{g} \oplus \mathfrak{g}'$, i.e. $\mathcal{U}(\mathfrak{g} \oplus \mathfrak{g}')\mathcal{H} = \mathcal{F}$.*

3. *Suppose $\pi \in \mathcal{R}(\mathfrak{g}, \widetilde{K}, \psi)$ and σ is a $\widetilde{K}$–type of π of minimal degree (Definition 6.2). Then $\sigma \in \mathcal{R}(K, \mathcal{H})$.*

4. *In the setting of (3), let $\pi' = \theta(\psi)(\pi)$, and let $\sigma' = \theta(\psi, \mathcal{H})(\sigma)$. Then σ' is a $\widetilde{K}'$–type of π' of lowest degree.*

Thus the duality correspondence θ comes equipped with a correspondence $\theta(\psi, \mathcal{H})$ of $\widetilde{K}, \widetilde{K}'$–types of lowest degree. It is therefore important to understand $\theta(\psi, \mathcal{H})$. From the results of Section 5 this may be computed explicitly, as we now describe.

We continue to work with our dual pair (G, G') together with two auxiliary pairs (K, M') and (M, K'). We also consider the maximal compact subgroups K_M and $K_{M'}$ of M and M'; $(K_M, K_{M'})$ is a dual pair with both members compact. We have the diagram:

$$\begin{array}{ccccc} & & M \longleftrightarrow K' & & \\ & \nearrow \uparrow & & \downarrow \searrow & \\ K_M & & G \longleftrightarrow G' & & K_{M'} \\ & \nwarrow \uparrow & & \downarrow \swarrow & \\ & & K \longleftrightarrow M' & & \end{array}$$

In the presence of several dual pairs we will write $\theta(\psi, K, M')$ for the duality correspondence between representations of $\widetilde{K}$ and $\widetilde{M}'$ via the dual pair (K, M'), and others similarly.

Lemma 7.4 ([21], Lemma 4.1) *Let σ be a $\widetilde{K}$–type of degree d. Define:*

1. *$\pi' = \theta(\psi, K, M')(\sigma)$,*

2. τ' *the unique* $\widetilde{K}_{M'}$*–type of* π' *of degree* d*,*

3. $\tau = \theta(\psi, K_{M'}, K_M)(\tau')$.

Then

1. *Suppose* τ *is the* $\widetilde{K}_M$*–type of lowest degree in a representation* π *of* M *with* $\sigma' = \theta(\psi, M, K')(\pi) \neq 0$*. Then* $\sigma\check{\otimes}\sigma'$ *occurs in* $\mathcal{H}$*, i.e.* $\theta(\psi, \mathcal{H})(\sigma) = \sigma'$*. The degrees of* σ, σ', τ *and* τ' *are all* d*. Furthermore all other* $\widetilde{K}'$*–types of* τ' *restricted to* $\widetilde{K}'$ *have degree strictly less than* d*.*

2. *Suppose condition (1) fails. Then* σ *does not occur in* $\mathcal{H}$*, i.e.* $\theta(\psi, \sigma) = 0$*, and all* $\widetilde{K}'$*–types of* τ' *restricted to* $\widetilde{K}'$ *have degree strictly less than* d*.*

This is illustrated by the following diagram.

$$\begin{array}{ccccccccccccc} K & \overset{\theta(K,M')}{\longrightarrow} & M' & \overset{\text{min. deg.}}{\longrightarrow} & K_{M'} & \overset{\theta(K_{M'},K_M)}{\longrightarrow} & K_M & \overset{\text{min. deg.}}{\longrightarrow} & M & \overset{\theta(M,K')}{\longrightarrow} & K' \\ \sigma & \longrightarrow & \pi' & \longrightarrow & \tau' & \longrightarrow & \tau & \longrightarrow & \pi & \longrightarrow & \sigma' \end{array}$$

The following explicit computation of the correspondence of joint harmonics is an immediate consequence of this Lemma and the formulas of Section 6. Normalizations and parameterizations of representations in terms of highest weights are as in Section 6.

Proposition 7.5 *Explicit correspondence of joint harmonics for irreducible dual pairs.*

$\theta(\psi, \mathcal{H})(\sigma) = \sigma'$ *for* σ *and* σ' *as follows.*

I. $(G, G') = (O(p, q), Sp(2n, \mathbb{R}))$, $(K, K') = (O(p) \times O(q), U(n))$.

$$\sigma = (a_1, \ldots, a_k, 0, \ldots, 0; \delta)\check{\otimes}(b_1, \ldots, b_\ell, 0, \ldots, 0; \epsilon) \rightarrow$$

$$\sigma' = (a_1, \ldots, a_k, \overbrace{1, \ldots, 1}^{\frac{1-\delta}{2}(p-2k)}, 0, \ldots, 0, \overbrace{-1, \ldots, -1}^{\frac{1-\epsilon}{2}(q-2\ell)}, -b_\ell, \ldots, -b_1) + (\frac{p-q}{2}, \ldots, \frac{p-q}{2}).$$

where $k + \frac{1-\delta}{2}(p-2k) + \ell + \frac{1-\epsilon}{2}(q-2\ell) \leq n$*.*

II. $(G, G') = (U(p, q), U(r, s))$, $(K, K') = (U(p) \times U(q), U(r) \times U(s))$.

$$\sigma = (a_1, \ldots, a_k, 0, \ldots, 0, b_1, \ldots, b_\ell; c_1, \ldots, c_m, 0, \ldots, 0, d_1, \ldots, d_n) + \frac{1}{2}(r-s, \ldots, r-s; s-r, \ldots, s-r) \to$$
$$\sigma' = (a_1, \ldots, a_k, 0, \ldots, 0, d_1, \ldots, d_n; c_1, \ldots, c_m, 0, \ldots, 0, b_1, \ldots, b_\ell) + \frac{1}{2}(p-q, \ldots, p-q; q-p, \ldots, q-p)$$

where the obvious inequalities hold:

$$k + \ell \le p, \; m + n \le q, \; k + n \le r, \; m + \ell \le s$$

III. $(G, G') = (Sp(p,q), O^*(2n))$, $(K, K') = (Sp(p) \times Sp(q), U(n))$.

(7.6)
$$(a_1, \ldots, a_r, 0, \ldots, 0; b_1, \ldots, b_s, 0, \ldots, 0) \to (a_1, \ldots, a_r, 0, \ldots, 0, -b_1, \ldots, -b_s) + (p-q, \ldots, p-q)$$

where $a_1 \ge \cdots \ge a_r > 0, b_1 \ge \cdots \ge b_s > 0$*,* $r \le p, s \le q, r + s \le n$*.*

IV. [33] $(G, G') = (GL(m, \mathbb{R}), GL(n, \mathbb{R}))$, $(K, K') = (O(m), O(n))$*. We assume* $m \le n$*.*

(7.7)
$$(a_1, \ldots, a_k, 0, \ldots, 0; \epsilon) \to \begin{cases} (a_1, \ldots, a_k, \overbrace{1, \ldots, 1}^{\frac{1-\epsilon}{2}(m-2k)}, 0, \ldots, 0; +) & k + \frac{1-\epsilon}{2}(m-2k) \le [\frac{n}{2}] \\ (a_1, \ldots, a_k, \overbrace{1, \ldots, 1}^{n-m}, 0, \ldots, 0; -) & else \end{cases}$$

where $a_1 \ge \cdots \ge a_k > 0$*,* $k \le [\frac{m}{2}]$*.*

V. [3] $(G, G') = (GL(m, \mathbb{C}), GL(n, \mathbb{C}))$, $(K, K') = (U(m), U(n))$.

(7.8)
$$(a_1, \ldots, a_k, 0, \ldots, 0, b_1, \ldots, b_\ell) \to (-b_\ell, \ldots, -b_1, 0, \ldots, 0, -a_k, \ldots, -a_1)$$

where $a_1 \ge \cdots \ge a_k > 0 > b_1 \ge \cdots \ge b_\ell$*,* $k + \ell \le m, n$*.*

VI. [3] $(G,G')=(O(m,\mathbb{C}),Sp(2n,\mathbb{C}))$, $(K,K')=(O(m),Sp(n))$.

$$(a_1,\dots,a_k,0,\dots,0;\epsilon)\to(a_1,\dots,a_k,\overbrace{1,\dots,1}^{\frac{1-\epsilon}{2}(m-2k)},0,\dots,0) \tag{7.9}$$

where $a_1\geq\dots a_k>0$, $k\leq[\frac{m}{2}]$, $k+\frac{1-\epsilon}{2}(m-2k)\leq n$.

VII. [32] $(G,G')=(GL(m,\mathbb{H}),GL(n,\mathbb{H}))$, $(K,K')=(Sp(m),Sp(n))$.

$$(a_1,\dots,a_k,0,\dots,0)\to(a_1,\dots,a_k,0,\dots,0) \tag{7.10}$$

where $a_1\geq\dots\geq a_k>0$ *and* $k\leq m,n$.

As an example we sketch one of the calculations involved in the proof of this Proposition.

Example 6.8 Let $(G,G')=(O(p,q),Sp(2n,\mathbb{R}))$. Then

$$(K,M')=(O(p)\times O(q),Sp(2n,\mathbb{R})\times Sp(2n,\mathbb{R}))$$

and $(M,K')=(U(p,q),U(n))$. Finally

$$(K_M,K_{M'})=(U(p)\times U(q),U(n)\times U(n)).$$

The embeddings $Sp(2n,\mathbb{R})\hookrightarrow Sp(2n,\mathbb{R})\times Sp(2n,\mathbb{R})$ and $U(n)\hookrightarrow U(n)\times U(n)$ are diagonal, and the restriction of an outer tensor product $\pi\check{\otimes}\pi'$ is the ordinary tensor product.

Let $\sigma=(a_1,\dots,a_k,0,\dots,0;\delta)\check{\otimes}(b_1,\dots,b_\ell,0,\dots,0;\epsilon)$. Then provided

$$k+\frac{1-\delta}{2}(p-2k)\leq n\ell \quad +\frac{1-\epsilon}{2}(q-2\ell)\leq n$$

we have

$$\tau'=(a_1+\frac{p}{2},\dots,a_k+\frac{p}{2},\overbrace{\frac{p}{2}+1,\dots,\frac{p}{2}+1}^{\frac{1-\delta}{2}(p-2k)},\frac{p}{2},\dots,\frac{p}{2})\check{\otimes}$$
$$(-\frac{q}{2},\dots,-\frac{q}{2},\overbrace{-\frac{q}{2}-1,\dots,-\frac{q}{2}-1}^{\frac{1-\epsilon}{2}(q-2\ell)},-b_\ell-\frac{q}{2},\dots,-b_1-\frac{q}{2})$$

Then

$$\tau = (a_1 + \frac{n}{2}, \dots, a_k + \frac{n}{2}, \overbrace{1 + \frac{n}{2}, \dots, 1 + \frac{n}{2}}^{\frac{1-\delta}{2}(p-2k)}, \frac{n}{2}, \dots, \frac{n}{2}) \check{\otimes}$$
$$(-\frac{n}{2}, \dots, -\frac{n}{2}, \overbrace{-1 - \frac{n}{2}, -1 - \frac{n}{2}}^{\frac{1-\epsilon}{2}(q-2\ell)}, -b_\ell - \frac{n}{2}, \dots, -b_1 - \frac{n}{2})$$

Finally σ' exists if and only if

$$k + \frac{1-\delta}{2}(p-2k) + \ell + \frac{1-\epsilon}{2}(q-2\ell) \leq n$$

in which case

$$\sigma' = (a_1 + \frac{p-q}{2}, \dots, a_k + \frac{p-q}{2}, \overbrace{1 + \frac{p-q}{2}, \dots, 1 + \frac{p-q}{2}}^{\frac{1-\delta}{2}(p-2k)}, \frac{p-q}{2}, \dots, \frac{p-q}{2},$$
$$\overbrace{-1 + \frac{p-q}{2}, \dots, -1 + \frac{p-q}{2}}^{\frac{1-\epsilon}{2}(q-2\ell)}, -b_\ell + \frac{p-q}{2}, \dots, -b_1 + \frac{p-q}{2})$$

Therefore $\mathcal{H} \simeq \sum \sigma \check{\otimes} \sigma'$ where the sum runs over σ, σ' satisfying (6.9)(d).

8 Induction principle

The discussion in this section is closely related to that of section II.6 of Roberts' second lecture [42]. The essential point is the computation of the Jacquet module of the oscillator representation with respect to the nilpotent radical of a parabolic subgroup of a dual pair. This was originally discussed in [40] in somewhat different terms, and first expressed this way in [29] for orthogonal/symplectic pairs over p–adic fields. This was generalized in [34] to all type I pairs. The case of the real dual pairs $(O(2p, 2q), Sp(2n, \mathbb{R}))$ is treated in [33], general real symplectic pairs in [4], and complex groups in [3].

We begin with a rough discussion of the principle. For the moment we ignore the covering groups, and write $G{\cdot}G' = G \times G'$, etc. Consider a dual pair (G, G') in some $Sp(2n, \mathbb{R})$ and parabolic subgroups $P = MN$ and $P' = M'N'$

of G and G' respectively. We assume, as is often the case, that (M, M') is itself a dual pair in some $Sp(2m, \mathbb{R})$. Let ω be an oscillator representation for $\widetilde{Sp}(2n, \mathbb{R})$ restricted to $G \times G'$, and let ω_M be an oscillator representation for $M \times M'$. The Jacquet module $\omega_{N \times N'}$ of ω is a representation of $M \times M'$, and a calculation shows it has $\omega_M \zeta$ as a quotient for some character ζ of $M \times M'$. Suppose there is a non–zero $M \times M'$ map

$$\phi : \omega_M \to \sigma \check{\otimes} \sigma'$$

for some representation $\sigma \otimes \sigma'$ of $M \times M'$. That is $\theta(\psi, M, M')(\sigma) = \sigma'$. It follows from the computation of the Jacquet module that there is a non–zero $G \times G'$–map

$$\Phi : \omega \to Ind_P^G(\sigma\xi) \check{\otimes} Ind_{P'}^{G'}(\sigma'\xi')$$

for some characters ξ, ξ'. That is $\theta(\psi, G, G')(\pi) = \pi'$ for some irreducible constituents π and π' of $Ind_P^G(\sigma\xi)$ and $Ind_{P'}^{G'}(\sigma'\xi')$ respectively.

This produces a large part of the correspondence for (G, G'). The problem, in general, is to determine the constituents of the induced modules in the image of Φ.

The same discussion holds in the p–adic case as well. In the real case, one has the extra very powerful information on K–types, and this is the primary tool for attacking the question. For example if π and a K–type σ of lowest degree in π are known, this determines a K'–type $\theta(\psi, \mathcal{H})(\sigma)$ in π', and this may be enough to describe π' explicitly.

Another important difference from the p–adic case is that the Jacquet–functor is exact over a p–adic field but not over $\mathbb{R}$, so the precise information about the filtration of the Jacquet module of [29],[34] is not available over $\mathbb{R}$. At the same time there is no notion of "supercuspidal" representation over $\mathbb{R}$, so the whole notion of "first occurrence" (cf. [42]) takes on a somewhat different form.

We turn now to a more careful description. We consider only type I dual pairs, so we are given a division algebra D of dimension $d = 1, 2$ or 4 over $\mathbb{R}$ with involution (cf. Section 5). We are also given V (resp. W) a vector space over D equipped with a non–degenerate Hermitian (resp. skew–Hermitian) form and $G = U(V)$ is the isometry group of this form (resp. $G' = U(W)$.) Then (G, G') is an irreducible reductive dual pair in $Sp(\mathbb{W}) \simeq Sp(2dn, \mathbb{R})$, where $\mathbb{W} = V \otimes_D W$ considered as a real vector space.

We consider an orthogonal direct sum

$$V = V_+ \oplus V^0 \oplus V_-$$

Here the form restricted to V^0 is non–degenerate, V_+ and V_- are isotropic, and the form defines a perfect pairing on $V_+ \times V_-$. The stabilizer $P = MN$ of V_- in G is a parabolic subgroup of G, with $M \simeq GL(D, V_+) \times U(V^0)$.

Similarly we consider

$$W = W_+ \oplus W^0 \oplus W_-$$

with $P' = M'N'$, and $M' \simeq GL(D, W_+) \times U(W^0)$.

Fix ψ and let ω be the corresponding (smooth) oscillator representation of $\widetilde{Sp}(2dn, \mathbb{R})$. We consider this as a representation of $\widetilde{G} \cdot \widetilde{G}'$ by restriction. Let $\mathcal{F}$ be the corresponding Fock model, considered as a $(\mathfrak{g} \oplus \mathfrak{g}', \widetilde{K} \cdot \widetilde{K}')$–module.

Now (M, M') is a dual pair in $Sp(\mathbb{W}_M)$ where $\mathbb{W}_M = (V_+ \otimes_D W_+) \oplus (V_- \otimes_D W_-) \oplus (V^0 \otimes W^0)$, considered as a real vector space. Let ω_M be the oscillator representation of this metaplectic group, with corresponding Fock model $\mathcal{F}_M$. A technical headache is that the inverse images of M and M' in $\widetilde{Sp}(\mathbb{W}_M)$ may not be isomorphic to $\widetilde{M}$ and $\widetilde{M}'$, so denote these covers $\overline{M}, \overline{M}'$. We consider ω_M as a representation of $\overline{M} \cdot \overline{M}'$, and $\mathcal{F}_M$ as a $(\mathfrak{m} \oplus \mathfrak{m}', \overline{K}_M \overline{K}_{M'})$–module (with the obvious notation for maximal compact subgroups of M and M' and their covers).

Let $\widetilde{P} \simeq \widetilde{M} N$ and $\widetilde{P}' \simeq \widetilde{M}' N'$ be the inverse images of P and P' in $Sp(\mathbb{W})$.

Theorem 8.1

1. *There is a non–zero surjective $\widetilde{P} \cdot \widetilde{P}'$ equivariant map*

$$\Phi : \omega \twoheadrightarrow \omega_M \chi$$

where χ is some character of $\overline{M} \cdot \overline{M}'$. Here $\overline{P} \cdot \overline{P}'$ acts on the right hand side by the given action of $\overline{M} \cdot \overline{M}'$, with $N \cdot N'$ acting trivially.

2. *There is a non–zero surjective $(\mathfrak{m} \oplus \mathfrak{m}', \widetilde{K}_M \cdot \widetilde{K}_{M'})$–map*

$$\Phi : \mathcal{F} \twoheadrightarrow \mathcal{F}_M \chi.$$

Here $\mathcal{F}$ may be replaced by the Lie algebra homology group $H_0(\mathfrak{n} \oplus \mathfrak{n}', \mathcal{F})$.

3. *Suppose σ (resp. σ') is an irreducible $(\mathfrak{m}, \overline{K}_M)$ (resp. $(\mathfrak{m}', \overline{K}_{M'})$) module, and σ corresponds to σ' in the duality correspondence for the dual pair (M, M'). Then there is a non–zero map*

$$\Phi : \omega \to Ind_{\widetilde{P}}^{\widetilde{G}}(\sigma\zeta)\check{\otimes}Ind_{\widetilde{P}'}^{\widetilde{G}'}(\sigma'\zeta')$$

where ζ, ζ' are certain characters of $\widetilde{M}$ and $\widetilde{M}'$.

4. *In the setting of (3), some irreducible constituents of*

$$Ind_{\widetilde{P}}^{\widetilde{G}}(\sigma\zeta)$$

and

$$Ind_{\widetilde{P}'}^{\widetilde{G}'}(\sigma'\zeta')$$

correspond via the duality correspondence for (G, G').

Note: This essentially follows from the computation of the top term of the filtration of the Jacquet module [29],[34] applied twice, once each to $P \subset G$ and $P' \subset G'$.

Note: Some covering problems are being swept under the rug. The main point is that σ and ζ are representations of $\overline{M}$, but that $\sigma\zeta$ may be identified with a representation of $\widetilde{M}$. In the orthogonal–symplectic case this is written out in in [4]. Similar but not identical covering issues are treated in the p–adic case in [34]. In [33] the same issues are addressed in a situation where the covering groups are all trivial.

Proof. The main point is to choose the proper (mixed) model of ω. For a complete polarization of $\mathbb{W}$ we take

$$X = (V \otimes_D W_+) \oplus (V_+ \otimes_D W^0) \oplus X^0$$
$$Y = (V \otimes_D W_-) \oplus (V_- \otimes_D W^0) \oplus Y^0$$

where X_0, Y_0 is an arbitrary complete polarization of $V^0 \otimes_D W^0$. Then ω is realized on the Schwarz space $\mathcal{S}(Y)$. For ω_M we take

$$X_M = (V_- \otimes_D W_+) \oplus X^0$$
$$Y_M = (V_+ \otimes_D W_-) \oplus Y^0$$

so ω_M is realized on $\mathcal{S}(Y_M)$.

An explicit calculation shows that restriction from Y to Y_M intertwines the action of $\widetilde{P}, \widetilde{P}'$ up to certain characters, which gives (1).

The corresponding statement for the Fock models follows from Lemma 4.1. In fact it follows that up to terms of lower degree, $\Phi : \mathcal{F} \twoheadrightarrow \mathcal{F}_M$ is given by restriction.

Statement (3) is an immediate consequence of Frobenius reciprocity [52, Proposition 6.3.5]. The final statement follows from (3) by composing Φ with the map from the image of Φ in the induced representation to an irreducible quotient of this image. □

We note that in this generality there is very little that can be said about the image of Φ and an irreducible quotient of it. Even if the induced representation has a unique irreducible quotient, there is no reason *a priori* that it should be in the image of Φ.

Open problem:

Prove a version of Theorem 8.2 for cohomological induction.

It appears that some version of Theorem 8.2 should hold with parabolic induction from M replaced by cohomological induction from the Levi factor L of a theta–stable parabolic [52],[26]. This is true in many examples, and some calculations indicate it is true in some generality. The homology groups that enter into Frobenius reciprocity in this setting ([52, Proposition 6.3.2]) are not solely in degree zero. As a result in the analogue of Theorem 8.2(2) it is necessary to calculate some higher homology groups.

Such a theorem together with Theorem 8.2 would go very far towards a complete explicit understanding of the duality correspondence.

9 Examples

We give a few examples of the local theta–correspondence over $\mathbb{R}$.

Example 1. $(O(1), Sp(2n, \mathbb{R}))$.

The center of $Sp(2n, \mathbb{R})$ is $O(1) \simeq \pm 1$, and this forms a dual pair. The inverse image $\widetilde{O}(1)$ of $O(1)$ in $\widetilde{Sp}(2n, \mathbb{R})$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ (n odd) or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (n even). This group has two genuine characters χ and χ' which correspond to the two irreducible summands of the oscillator representation ω_ψ. The labeling of χ and and χ' is a matter of convention, or equivalently of a choice of isomorphism $\widetilde{O}(1) \simeq \mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The same two characters also give the two constituents of $\omega_{\overline{\psi}}$.

Example 2. $(U(1), U(n))$.

Here $U(1)$ is the center of the maximal compact subgroup $K = U(n)$. This dual pair describes the restriction of the oscillator representation to $\widetilde{K}$, which is quite important.

Fix ψ. With choices as in Section 6, example II (preceding Proposition 5.6) the correspondence from $U(1)$ to $U(n)$ is $(k+\frac{n}{2}) \to (k+\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, $k = 0, 1, 2, \ldots$. The $\widetilde{K}$–types with k even (resp. odd) constitute the irreducible summand ω_ψ^+ (resp. ω_ψ^-). The lowest $\widetilde{K}$–types of these two summands (both in the sense of Vogan, and of lowest degree) are $(\frac{1}{2}, \ldots, \frac{1}{2})$ and $(\frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ respectively.

Note that the $\widetilde{K}$–types lie along a "line", i.e. their highest weights are obtained from the highest weight of the lowest $\widetilde{K}$–type by adding multiples of a single vector. This is a condition of [53]. In fact the four irreducible summands of the oscillator representations are the only non–trivial unitary representations of $\widetilde{Sp}(2n, \mathbb{R})$ $(n \geq 1)$ with $\widetilde{K}$–types along a line. The oscillator representation is particularly "small", and the duality correspondence is due in part to this. It is interesting to note that it is necessary to pass to the two fold cover of $Sp(2n, \mathbb{R})$ to find these especially small representations. This is analogous to the spin representations of the two-fold cover $Spin(n)$ of $SO(n)$.

Once ψ is fixed, the dual oscillator representation $\omega_\psi^* = \omega_{\overline{\psi}}$ has $\widetilde{K}$–spectrum $(-\frac{1}{2}, \ldots, -\frac{1}{2}, -\frac{1}{2} - k)$ $(k = 0, 1, \ldots)$.

Example 3. $(O(n), SL(2, \mathbb{R}))$. The duality correspondence in this case is essentially the classical theory of spherical harmonics [20],[23]. With the appropriate coordinates the Fock model is on $\mathbb{C}[z_1, \ldots, z_n]$. One of the operators coming from the (complexified) Lie algebra of $SL(2, \mathbb{R})$ is the Laplacian $\Delta = \sum_i \frac{d^2}{dz_i^2}$, and the harmonics in the sense of section 5 are the kernel of the Laplacian. Then $r^2 = \sum_i z_i^2$ and $\sum z_i \frac{d}{dz_i} + \frac{n}{2}$ together with Δ span the image of $\mathfrak{sl}(2, \mathbb{C})$. Theorem 6.3 (Theorem 9 of [20]) reduces to the classical statements of spherical harmonics in this case.

Example 4. $(GL(1, \mathbb{R}), GL(1, \mathbb{R}))$

This example is an illustration of the principle that no dual pair is too simple to be taken lightly. The p–adic case is treated in [34, Chapter 3,

§III.7, pg. 65] and has some non–trivial analytic content. The corresponding statement over $\mathbb{R}$ for the smooth vectors is similar. Here we discuss only the Fock model.

Write the dual pair as (G_1, G_2); of course the images of G_1 and G_2 in $SL(2,\mathbb{R})$ coincide. We embed $G_1 \hookrightarrow SL(2,\mathbb{R})$ as $x \to \iota_1(x) = diag(x, x^{-1})$. Write $\widetilde{SL}(2,\mathbb{R})$ a as $\{(g, \pm 1) \mid g \in SL(2,\mathbb{R})\}$ with the usual cocycle [41]. Let $\widetilde{GL}(1,\mathbb{R})$ be the two–fold cover of $GL(1,\mathbb{R})$ defined by the cocycle $c(x,y) = (x,y)_{\mathbb{R}}$. Here $(x,y)_{\mathbb{R}}$ is the Hilbert symbol [46], which equals -1 if $x, y < 0$, and 1 otherwise. This is isomorphic to the $det^{\frac{1}{2}}$ cover, or alternatively to $\mathbb{R}^\times \cup i\mathbb{R}^\times \subset \mathbb{C}^\times$. Then $\widetilde{\iota_1} : (x,\epsilon) \to (\iota(x), \epsilon)$ gives an isomorphism of $\widetilde{GL}(1,\mathbb{R})$ with the inverse image of $GL(1)$ in $\widetilde{SL}(2,\mathbb{R})$. Let $\mathfrak{g}_1 \simeq \mathbb{C}$ be the complexified Lie algebra of G_1, and let $K_1 = \pm 1$ be its maximal compact subgroup, so $\widetilde{K}_1 \simeq \mathbb{Z}/4\mathbb{Z}$.

Fix ψ and the Fock model $\mathcal{F} = \mathbb{C}[z]$. We seek to show that every genuine irreducible $(\mathfrak{g}_1, \widetilde{K}_1)$–module is a quotient of $\mathcal{F}$. The action of $\widetilde{K}_1$ breaks up $\mathcal{F}$ into its two irreducible summands $\mathcal{F}^+ = \mathbb{C}[z^2]$ and $\mathcal{F}^- = z\mathbb{C}[z^2]$ for $\widetilde{SL}(2,\mathbb{R})$ (cf. Example 1). So it is enough to show any character of $\mathfrak{g}_1$ occurs as a quotient of one summand.

The action of the Lie algebra $\mathfrak{g}_1$ of $GL(1,\mathbb{R})$ is by the operator $X = z^2 - \frac{d^2}{dz^2}$ (cf. Section 4). Therefore $\mathcal{F}^\pm$ are each free modules for this action, and every character of $\mathfrak{g}_1$ occurs as a quotient of $\mathcal{F}^\pm$ in a unique way. To be explicit we consider $\mathcal{F}^+$. Fix $\lambda \in \mathbb{C} \simeq \mathfrak{g}_1^*$, with $\lambda(X) = \lambda$, and let $\mathcal{N}_\lambda = \mathcal{U}(\mathfrak{g}) \cdot (z^2 - \lambda)$. By induction this is a codimension one subspace of $\mathcal{F}^+$. The image $\overline{1}$ of 1 in the quotient $\mathcal{F}^+/\mathcal{N}_\lambda$ is non–zero, and $X \cdot \overline{1} = \overline{z^2} = \lambda\overline{1}$. The case of $\mathcal{F}^-$ is similar.

The embedding of the second copy G_2 of $GL(1)$ is $\iota_2 : x = diag(x^{-1}, x)$. The natural choice of $\widetilde{\iota_2}$ is $(x,\epsilon) \to (\iota_2(x), \epsilon))$. With this convention $\widetilde{\iota_2}^{-1} \circ \widetilde{\iota_1} : \widetilde{G}_1 \to \widetilde{G}_2$ takes (x,ϵ) to $(x^{-1}, \epsilon) = (x, \epsilon sgn(x))^{-1}$.

With these choices, the duality correspondence is

$$\chi \to \chi^{-1} sgn$$

for any genuine $(\mathfrak{g}_1, \widetilde{K}_1)$ character χ. The sgn term comes from the twist by $sgn(x)$ in $\iota_2^{-1} \circ \iota_1$.

Of course $\widetilde{\iota}_2$ can be modified to eliminate the twist by sgn. Also the oscillator representation itself restricted to this dual pair can be normalized by tensoring with a genuine character of $\widetilde{GL}(1,\mathbb{R})$ so that the correspondence takes $\chi \to \chi^{-1}$ as χ runs over characters of $GL(1,\mathbb{R})$. This is what is normally done.

Example 5. $(GL(m,\mathbb{R}), GL(m,\mathbb{R}))$.

This generalizes Example 4. Let G be the $det^{\frac{m}{2}}$ cover of $GL(m,\mathbb{R})$. Then (G_1, G_2) is a dual pair with $G_1 \simeq G \simeq G_2$, and with the natural choice of these isomorphisms the duality correspondence takes π to $\pi^* \otimes sgn$, as π runs over all irreducible genuine representations of G. As in Example 4 the twist by sgn can be eliminated, and G replaced by $GL(m,\mathbb{R})$.

The proof is by induction on m, starting with $m = 1$ by Example 4, and the induction principle of Section 8. We normalize the correspondence to eliminate the covering groups. We write an irreducible representation π of $GL(m)$ as the unique irreducible quotient of

$$Ind_{MN}^{G_1}(\sigma \otimes \mathbb{1})$$

where $M \simeq GL(m-1) \times GL(1)$. By induction and Theorem 8.2 there is a non–zero map from ω to the tensor product of the induced module for G_1 and a similar induced modules for G_2 with σ replaced by σ^*. If these induced modules are irreducible the result is immediate. The general case follows from a deformation of parameters argument as in [3], the main point being that we have enough control over K–types to determine at least an irreducible quotient of the image of this map. The K–type information is crucial because the induced module for G_2 will have unique irreducible submodule, and not a quotient. This information is not available in the p–adic case, and these elementary techniques are not enough to determine the correspondence in this case (cf. [34]).

Example 6. $(GL(m,\mathbb{R}), GL(n,\mathbb{R}))$ Based on 5 the general case of $GL(m,\mathbb{R})$ is straightforward [33]. We normalize the correspondence and eliminate the covering groups. Suppose $m \leq n$. Then every representation π of $GL(m,\mathbb{R})$ occurs in the correspondence. Let $MN \simeq GL(m) \times GL(n-m) \times N$ be the usual maximal parabolic subgroup of $GL(n)$. Then $\theta(\pi)$ is the is the unique irreducible constituent of

$$Ind_{MN}^{GL(n)}(\pi \otimes \mathbb{1} \otimes \mathbb{1})$$

containing a certain K–type of multiplicity one.

A similar result holds for $(GL(m,\mathbb{C}), GL(n,\mathbb{C}))$ [3], and also for the dual pairs $(GL(m,\mathbb{H}), GL(n,\mathbb{H}))$ [32].

Example 7. $(O(m,\mathbb{C}), Sp(2n,\mathbb{C}))$

We parameterize irreducible representations of complex groups by pairs (λ,ν) as in [35]. We fix an orthogonal group $G_1 = O(2m+\tau,\mathbb{C})$ $(\tau = 0,1)$ and consider the family of dual pairs $(G_1, G_2(n))$ with $G_2(n) = Sp(2n,\mathbb{C})$. Given an irreducible representation π_1 of G_1, there exists a non–negative integer $n(\pi_1)$ such that π occurs in the duality correspondence for $(G_1, G_2(n))$ if and only if $n \geq n(\pi_1)$. In the p–adic case this is covered in the lectures of Brooks Roberts [43], see also [33],[3].

For unexplained notation see [3].

Theorem 9.1 *Let $\pi_1 = L(\mu_1,\nu_1)$ be an irreducible representation of G_1.*

Define the integer $k = k[\mu_1]$ by writing $\mu_1 = (a_1,\dots,a_k,0,\dots,0;\epsilon)$ with $a_1 \geq a_2 \geq \dots \geq a_k > 0$. Write $\nu_1 = (b_1,\dots,b_m)$, and define the integer $0 \leq q = q[\mu_1,\nu_1] \leq m-k$ to be the largest integer such that $2q-2+\tau, 2q-4+\tau,\dots,\tau$ all occur (in any order) in $\{\pm b_{k+1}, \pm b_{k+2},\dots,\pm b_m\}$. After possibly conjugating by the stabilizer of μ_1 in W, we may write

$$\mu_1 = (\overbrace{a_1,\dots,a_k}^{k}, \overbrace{0,0,\dots,0}^{m-q-k}\overbrace{0,0,\dots,0}^{q}; \epsilon)$$

$$\nu_1 = (\overbrace{b_1,\dots,b_k}^{k}, \overbrace{b_{k+1},\dots,b_{m-q}}^{m-q-k}, \overbrace{2q-2+\tau, 2q-4+\tau,\dots,\tau}^{q}).$$

Let $\mu_1' = (a_1,\dots,a_k)$, $\nu_1' = (b_1,\dots,b_k)$, and $\nu_1'' = (b_{k+1},\dots,b_{m-q})$.

Then $n(\pi_1) = m - \epsilon q + \frac{1-\epsilon}{2}\tau$, and for $n \geq n(\pi_1)$, $\theta(\pi_1) = L(\mu_2,\nu_2)$, where

$$\mu_2 = (\mu_1', \overbrace{1,\dots,1}^{\frac{1-\epsilon}{2}(2q+\tau)}, 0,0,\dots,0)$$

$$\nu_2 = (\nu_1', \overbrace{2q-1+\tau, 2q-3+\tau,\dots,2\epsilon q+1+\epsilon\tau}^{\frac{1-\epsilon}{2}(2q+\tau)}, \nu_1'',$$
$$2n-2m-\tau, 2n-2m-2-\tau,\dots,-\epsilon(2q+\tau)+2).$$

Example 8. $(O(p,q), Sp(2n,\mathbb{R}))$ with $p+q = 2n, 2n+1, 2n+2$.

In these examples the groups are the same "size", and are of particular interest from the point of view of L–functions (cf. the lectures of Steve Kudla). This is the opposite extreme of the stable range [14]. The case $(O(2,2), Sp(4,\mathbb{R}))$ is in [38] ($O(4,0)$ and $O(0,4)$ are in [25], see Section 6.). The cases $(O(p,q), Sp(2n,\mathbb{R}))$ with $p+q = 2n, 2n+2$ and p,q even are in [33], p, q odd are only missing because of covering group technicalities. Finally $(O(p,q), Sp(2n\mathbb{R}))$ witih $p+q=2n+1$ is in [4], this is similar to [33] except that the covering groups are unavoidable.

We first consider the case p, q even. In this case the covering of $Sp(2n,\mathbb{R})$ splits and the correspondence can be written in terms of the linear groups. Roughly speaking the correspondence in these cases is "functorial", and a number of nice properties hold which fail in general. In particular the minimal K–type in the sense of Vogan is always of minimal degree in this situation.

The duality correspondence is described explicitly in terms of Langlands parameters, and these match up in a natural way. This can be expressed in terms of a homomorphism between the L–groups. The disconnectedness of $O(p,q)$ can be avoided in this range; at most one of π and $\pi \otimes sgn$ occur in the correspondence.

If $p+q=2n+2$ every representation of $Sp(2n,\mathbb{R})$ occurs in the correspondence with some $O(p,q)$ (perhaps more than one). If $p+q=2n$ every representation of $Sp(2n,\mathbb{R})$ occurs with at most one $O(p,q)$ (but some may fail to occur.) This suggests that $p+q=2n+1$ should be particularly nice, and this is the case. Fix $\delta=\pm 1$ and consider the dual pairs $(O(p,q), Sp(2n,\mathbb{R}))$ with $(-1)^q=\delta$ and $p+q=2n+1$. The covering of $Sp(2n,\mathbb{R})$ is the metaplectic group, and the representations which occur are all genuine. We twist by a genuine character of the cover of $O(p,q)$ to pass to representations of the linear group.

Fix ψ. Then every genuine representation of $\widetilde{Sp}(2n,\mathbb{R})$ occurs with precisely one $O(p,q)$, and of every pair of representations π and $\pi \otimes sgn$ of $O(p,q)$ (these are not isomorphic) precisely one occurs. By restricting to $SO(p,q)$, this establishes a bijection between the set

$$\widetilde{Sp}(2n,\mathbb{R})_{genuine}\widehat{\ }$$

of genuine irreducible admissible representations of $\widetilde{Sp}(2n,\mathbb{R})$ and the union

$$\bigcup_{p+q=2n+1 (-1)^q=\delta} SO(p,q)\widehat{\ }$$

of the irreducible admissible representations of the groups $SO(p,q)$.

The notion of functoriality is not well–defined for the non–linear group $\widetilde{Sp}(2n,\mathbb{R})$. Nevertheless this correspondence is "functorial" in some sense. It is naturally described in terms of Langlands parameters, and (Vogan) lowest

K–types are always of lowest degree. The orbit correspondence [14] is a bijection between the regular semisimple coadjoint orbits for $Sp(2n,\mathbb{R})$ and the union of those for $O(p,q)$. At least philosophically this underlies the correspondence. Again similar but not quite so clean results hold in the case $p+q=2n,2n+2$.

Example 10. Results of [33]

The dual pairs $(O(p,q),Sp(2n,\mathbb{R}))$ with p,q even are discussed extensively by Moeglin in [33]; the results on $p+q=2n,2n+2$ of the preceding example are a very special case.

Roughly speaking, Moeglin first considers discrete series representations which correspond to discrete series in a dual pair with (G_1,G_2) the same size. Theorem 8.2 then produces certain constituents of some induced modules which correspond. The idea is to use the K–type information of Section 6 to determine these constituents, including their Langlands parameters. This program works provided the (Vogan) lowest K–types and K–types of lowest degree coincide for the representations in question.

The result is an explicit description of the correspondence in cases in which it is functorial. The precise conditions under which this holds are in terms of the lowest K–types; they are quite technical in general, and Example 8 is the cleanest special case. The worst case from this point of view is $(O(p,q),Sp(2n,\mathbb{R}))$ with n very small compared to p,q and $p-q$ large. In this regard we mention that the case $(O(p,q),SL(2,\mathbb{R}))$ is thoroughly described in [16], which interprets results of [50]and [39] in these terms, and also discussed in [23].

Example 10. Representations with cohomology.

Unitary representations with $(\mathfrak{g},K)$–cohomology have been classified [54]. These have regular integral infinitesimal character, i.e. the same infinitesimal character as that of a finite–dimensional representation.

Let (G_1,G_2) be a dual pair, and let π be an irreducible representation of G_1 with non–zero $(\mathfrak{g},K)$–cohomology (possibly with coefficients). Also assume the infinitesimal character of $\pi'=\theta(\pi)$ is regular and integral. The explicit description of the correspondence in this case is due to Jian-Shu Li [31]. It turns out that π' is a discrete series representation. The special case when (G_1,G_2) are in the stable range [14] is in [2], and in somewhat greater generality in [33].

It is known that the representations with cohomology exhaust the unitary representations with regular integral infinitesimal character [45]. Therefore Li's result implies that the theta–correspondence preserves unitarity in the case of regular integral infinitesimal character.

References

[1] J. Adams. Duality for the spin representation. www.math.umd.edu/~jda/preprints/spinduality.dvi.

[2] J. Adams. L-functoriality for dual pairs. *Astérisque*, (171-172):85–129, 1989. Orbites unipotentes et représentations, II.

[3] Jeffrey Adams and Dan Barbasch. Reductive dual pair correspondence for complex groups. *J. Funct. Anal.*, 132(1):1–42, 1995.

[4] Jeffrey Adams and Dan Barbasch. Genuine representations of the metaplectic group. *Compositio Math.*, 113(1):23–66, 1998.

[5] John C. Baez. The octonions. *Bull. Amer. Math. Soc. (N.S.)*, 39(2):145–205 (electronic), 2002.

[6] V. Bargmann. On a Hilbert space of analytic functions and an associated integral transform. *Comm. Pure Appl. Math.*, 14:187–214, 1961.

[7] P. Cartier. *Quantum mechanical commutation relations and θ-functions*, volume 9 of *Proc. Symp. Pure Math.* American Math. Soc., Providence, Rhode Island, 1966.

[8] Mark G. Davidson, Thomas J. Enright, and Ronald J. Stanke. Differential operators and highest weight representations. *Mem. Amer. Math. Soc.*, 94(455):iv+102, 1991.

[9] Thomas Enright, Roger Howe, and Nolan Wallach. A classification of unitary highest weight modules. In *Representation theory of reductive groups (Park City, Utah, 1982)*, volume 40 of *Progr. Math.*, pages 97–143. Birkhäuser Boston, Boston, MA, 1983.

[10] Stephen Gelbart. Holomorphic discrete series for the real symplectic group. *Invent. Math.*, 19:49–58, 1973.

[11] Stephen Gelbart. Examples of dual reductive pairs. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 287–296. Amer. Math. Soc., Providence, R.I., 1979.

[12] Kenneth I. Gross and Ray A. Kunze. Bessel functions and representation theory. II. Holomorphic discrete series and metaplectic representations. *J. Functional Analysis*, 25(1):1–49, 1977.

[13] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 34 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Corrected reprint of the 1978 original.

[14] R. Howe. L^2-duality for stable dual pairs. preprint.

[15] R. Howe. θ-series and invariant theory. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 275–285. Amer. Math. Soc., Providence, R.I., 1979.

[16] Roger Howe. On some results of Strichartz and Rallis and Schiffman. *J. Funct. Anal.*, 32(3):297–303, 1979.

[17] Roger Howe. Quantum mechanics and partial differential equations. *J. Funct. Anal.*, 38(2):188–254, 1980.

[18] Roger Howe. Dual pairs in physics: harmonic oscillators, photons, electrons, and singletons. In *Applications of group theory in physics and mathematical physics (Chicago, 1982)*, volume 21 of *Lectures in Appl. Math.*, pages 179–207. Amer. Math. Soc., Providence, RI, 1985.

[19] Roger Howe. The oscillator semigroup. In *The mathematical heritage of Hermann Weyl (Durham, NC, 1987)*, volume 48 of *Proc. Sympos. Pure Math.*, pages 61–132. Amer. Math. Soc., Providence, RI, 1988.

[20] Roger Howe. Remarks on classical invariant theory. *Trans. Amer. Math. Soc.*, 313(2):539–570, 1989.

[21] Roger Howe. Transcending classical invariant theory. *J. Amer. Math. Soc.*, 2(3):535–552, 1989.

[22] Roger Howe. Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond. In *The Schur lectures (1992) (Tel Aviv)*, volume 8 of *Israel Math. Conf. Proc.*, pages 1–182. Bar-Ilan Univ., Ramat Gan, 1995.

[23] Roger Howe and Eng-Chye Tan. *Nonabelian harmonic analysis*. Universitext. Springer-Verlag, New York, 1992. Applications of $\mathrm{SL}(2,\mathbf{R})$.

[24] James E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1978. Second printing, revised.

[25] M. Kashiwara and M. Vergne. On the Segal-Shale-Weil representations and harmonic polynomials. *Invent. Math.*, 44(1):1–47, 1978.

[26] Anthony W. Knapp. *Lie groups, Lie algebras, and cohomology*, volume 34 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 1988.

[27] Anthony W. Knapp. *Representation theory of semisimple groups*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 2001. An overview based on examples, Reprint of the 1986 original.

[28] Stephen S. Kudla. Seesaw dual reductive pairs. In *Automorphic forms of several variables (Katata, 1983)*, volume 46 of *Progr. Math.*, pages 244–268. Birkhäuser Boston, Boston, MA, 1984.

[29] Stephen S. Kudla. On the local theta-correspondence. *Invent. Math.*, 83(2):229–255, 1986.

[30] Jian-Shu Li. Singular unitary representations of classical groups. *Invent. Math.*, 97(2):237–255, 1989.

[31] Jian-Shu Li. Theta lifting for unitary representations with nonzero cohomology. *Duke Math. J.*, 61(3):913–937, 1990.

[32] Jian-Shu Li, Annegret Paul, Eng-Chye Tan, and Chen-Bo Zhu. The explicit duality correspondence of $(\mathrm{Sp}(p,q), \mathrm{O}^*(2n))$. *J. Funct. Anal.*, 200(1):71–100, 2003.

[33] C. Mœglin. Correspondance de Howe pour les paires reductives duales: quelques calculs dans le cas archimédien. *J. Funct. Anal.*, 85(1):1–85, 1989.

[34] Colette Mœglin, Marie-France Vignéras, and Jean-Loup Waldspurger. *Correspondances de Howe sur un corps p-adique*, volume 1291 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987.

[35] K. R. Parthasarathy, R. Ranga Rao, and V. S. Varadarajan. Representations of complex semi-simple Lie groups and Lie algebras. *Ann. of Math. (2)*, 85:383–429, 1967.

[36] Dipendra Prasad. Weil representation, Howe duality, and the theta correspondence. In *Theta functions: from the classical to the modern*, volume 1 of *CRM Proc. Lecture Notes*, pages 105–127. Amer. Math. Soc., Providence, RI, 1993.

[37] Tomasz Przebinda. On Howe's duality theorem. *J. Funct. Anal.*, 81(1):160–183, 1988.

[38] Tomasz Przebinda. The oscillator duality correspondence for the pair $\mathrm{O}(2,2)$, $\mathrm{Sp}(2,\mathbf{R})$. *Mem. Amer. Math. Soc.*, 79(403):x+105, 1989.

[39] S. Rallis and G. Schiffmann. Représentations supercuspidales du groupe métaplectique. *J. Math. Kyoto Univ.*, 17(3):567–603, 1977.

[40] Stephen Rallis. Langlands' functoriality and the Weil representation. *Amer. J. Math.*, 104(3):469–515, 1982.

[41] R. Ranga Rao. On some explicit formulas in the theory of Weil representation. *Pacific J. Math.*, 157(2):335–371, 1993.

[42] B. Roberts. The theta correspondence and witt towers. www.math.umd.edu/∼jda/preprints/workshop_roberts2.dvi.

[43] B. Roberts. The weil representation and dual pairs. www.math.umd.edu/∼jda/preprints/workshop_roberts1.dvi.

[44] M. Saito. Représentations unitaires des groupes symplectiques. *Math. Soc. Japan*, 24:232–251, 1972.

[45] Susana A. Salamanca-Riba. On the unitary dual of real reductive Lie groups and the $A_g(\lambda)$ modules: the strongly regular case. *Duke Math. J.*, 96(3):521–546, 1999.

[46] J.-P. Serre. *A course in arithmetic*. Springer-Verlag, New York, 1973. Translated from the French, Graduate Texts in Mathematics, No. 7.

[47] David Shale. Linear symmetries of free boson fields. *Trans. Amer. Math. Soc.*, 103:149–167, 1962.

[48] M. J. Slupinski. Dual pairs in $\mathrm{Pin}(p, q)$ and Howe correspondences for the spin representation. *J. Algebra*, 202(2):512–540, 1998.

[49] Shlomo Sternberg. Some recent results on the metaplectic representation. In *Group theoretical methods in physics (Sixth Internat. Colloq., Tübingen, 1977)*, volume 79 of *Lecture Notes in Phys.*, pages 117–143. Springer, Berlin, 1978.

[50] Robert S. Strichartz. Harmonic analysis on hyperboloids. *J. Functional Analysis*, 12:341–383, 1973.

[51] Hans Tilgner. Graded generalizations of Weyland- and Clifford algebras. *J. Pure Appl. Algebra*, 10(2):163–168, 1977/78.

[52] David A. Vogan, Jr. *Representations of real reductive Lie groups*, volume 15 of *Progress in Mathematics*. Birkhäuser Boston, Mass., 1981.

[53] David A. Vogan, Jr. Singular unitary representations. In *Noncommutative harmonic analysis and Lie groups (Marseille, 1980)*, volume 880 of *Lecture Notes in Math.*, pages 506–535. Springer, Berlin, 1981.

[54] David A. Vogan, Jr. and Gregg J. Zuckerman. Unitary representations with nonzero cohomology. *Compositio Math.*, 53(1):51–90, 1984.

[55] Hermann Weyl. *The classical groups*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1997. Their invariants and representations, Fifteenth printing, Princeton Paperbacks.