Waveform design and Sigma-Delta quantization

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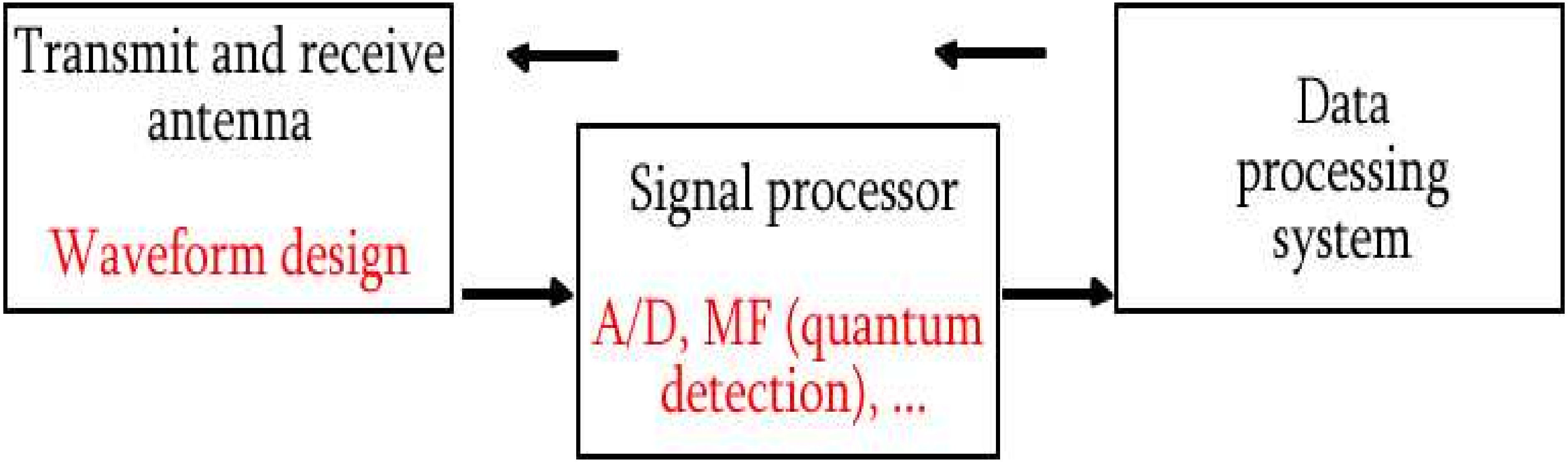
Outline and collaborators

1. CAZAC waveforms
2. Finite frames
3. Sigma-Delta quantization – theory and implementation
4. Sigma-Delta quantization – number theoretic estimates

Collaborators: Jeff Donatelli (waveform design); Matt Fickus (frame force); Alex Powell and Özgür Yilmaz ($\Sigma - \Delta$ quantization); Alex Powell, Aram Tangboondouangjit, and Özgür Yilmaz ($\Sigma - \Delta$ quantization and number theory).

Processing


Transmit and receive antenna
Waveform design
Signal processor
A/D, MF (quantum detection), ...
Data processing system

CAZAC waveforms

Definition of CAZAC waveforms

A K-periodic waveform $u : \mathbb{Z}_K = \{0, 1, \ldots, K-1\} \to \mathbb{C}$ is *Constant Amplitude Zero Autocorrelation (CAZAC)* if,

$$\text{for all } k \in \mathbb{Z}_K, |u[k]| = 1, \quad \text{(CA)}$$

and, for $m = 1, \ldots, K-1$, the *autocorrelation*

$$A_u[m] = \frac{1}{K} \sum_{k=0}^{K-1} u[m+k]\overline{u}[k] \text{ is } 0. \text{ (ZAC)}$$

The *crosscorrelation* of $u, v : \mathbb{Z}_K \to \mathbb{C}$ is

$$C_{u,v}[m] = \frac{1}{K} \sum_{k=0}^{K-1} u[m+k]\overline{v}[k]$$

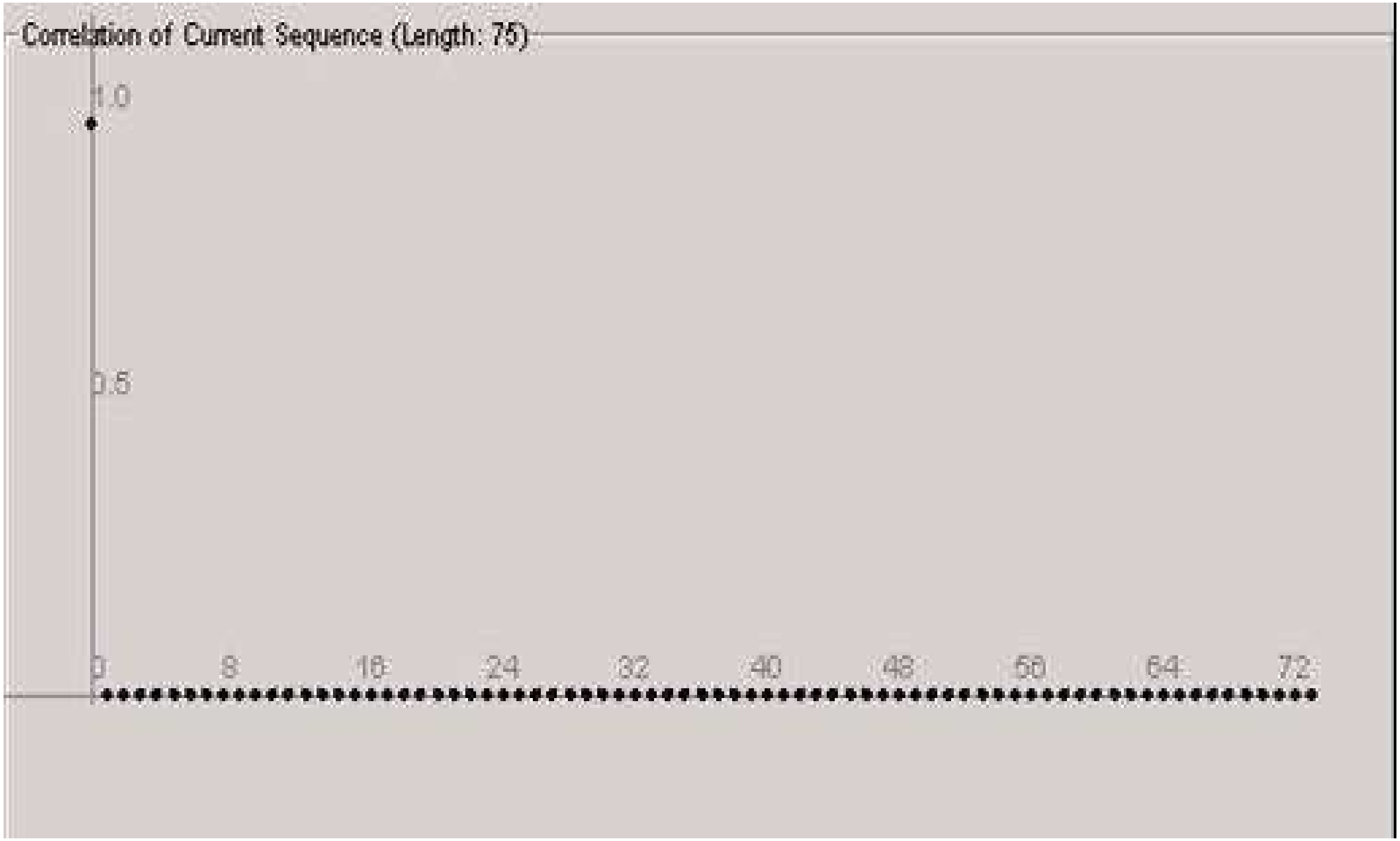
Properties of CAZAC waveforms

- u CAZAC $\Rightarrow$ u is broadband (full bandwidth).
- There are different constructions of different CAZAC waveforms resulting in different behavior vis à vis Doppler, additive noises, and approximation by bandlimited waveforms.
- u CA $\iff$ DFT of u is ZAC off dc. (DFT of u can have zeros)
- u CAZAC $\iff$ DFT of u is CAZAC.
- User friendly software: http://www.math.umd.edu/~jjb/cazac

Examples of CAZAC Waveforms

$K = 75 : u(x) =$
$(1, 1, 1, 1, 1, 1, e^{2\pi i\frac{1}{15}}, e^{2\pi i\frac{2}{15}}, e^{2\pi i\frac{1}{5}}, e^{2\pi i\frac{4}{15}}, e^{2\pi i\frac{1}{3}}, e^{2\pi i\frac{7}{15}}, e^{2\pi i\frac{3}{5}},$
$e^{2\pi i\frac{11}{15}}, e^{2\pi i\frac{13}{15}}, 1, e^{2\pi i\frac{1}{5}}, e^{2\pi i\frac{2}{5}}, e^{2\pi i\frac{3}{5}}, e^{2\pi i\frac{4}{5}}, 1, e^{2\pi i\frac{4}{15}}, e^{2\pi i\frac{8}{15}}, e^{2\pi i\frac{4}{5}},$
$e^{2\pi i\frac{16}{15}}, e^{2\pi i\frac{1}{3}}, e^{2\pi i\frac{2}{3}}, e^{2\pi i}, e^{2\pi i\frac{4}{3}}, e^{2\pi i\frac{5}{3}}, 1, e^{2\pi i\frac{2}{5}}, e^{2\pi i\frac{4}{5}}, e^{2\pi i\frac{6}{5}},$
$e^{2\pi i\frac{8}{5}}, 1, e^{2\pi i\frac{7}{15}}, e^{2\pi i\frac{14}{15}}, e^{2\pi i\frac{7}{5}}, e^{2\pi i\frac{28}{15}}, e^{2\pi i\frac{1}{3}}, e^{2\pi i\frac{13}{15}}, e^{2\pi i\frac{7}{5}}, e^{2\pi i\frac{29}{15}},$
$e^{2\pi i\frac{37}{15}}, 1, e^{2\pi i\frac{3}{5}}, e^{2\pi i\frac{6}{5}}, e^{2\pi i\frac{9}{5}}, e^{2\pi i\frac{12}{5}}, 1, e^{2\pi i\frac{2}{3}}, e^{2\pi i\frac{4}{3}}, e^{2\pi i\cdot 2}, e^{2\pi i\frac{8}{3}},$
$e^{2\pi i\frac{1}{3}}, e^{2\pi i\frac{16}{15}}, e^{2\pi i\frac{9}{5}}, e^{2\pi i\frac{38}{15}}, e^{2\pi i\frac{49}{15}}, 1, e^{2\pi i\frac{4}{5}}, e^{2\pi i\frac{8}{5}}, e^{2\pi i\frac{12}{5}}, e^{2\pi i\frac{16}{5}},$
$1, e^{2\pi i\frac{13}{15}}, e^{2\pi i\frac{26}{15}}, e^{2\pi i\frac{13}{5}}, e^{2\pi i\frac{52}{15}}, e^{2\pi i\frac{1}{3}}, e^{2\pi i\frac{19}{15}}, e^{2\pi i\frac{11}{5}}, e^{2\pi i\frac{47}{15}}, e^{2\pi i\frac{61}{15}})$

Autocorrelation of CAZAC $K = 75$



Perspective

Sequences for coding theory, cryptography, and communications (synchronization, fast start-up equalization, frequency hopping) include the following in the periodic case:

- Gauss, Wiener (1927), Zadoff (1963), Schroeder (1969), Chu (1972), Zhang and Golomb (1993)
- Frank (1953), Zadoff and Abourezk (1961), Heimiller (1961)
- Milewski (1983)
- Björck (1985) and Golomb (1992).

and their generalizations, both periodic and aperiodic, with some being equivalent in various cases.

Finite ambiguity function

Given K-periodic waveform, $u : \mathbb{Z}_K \to \mathbb{C}$ let $e_j[k] = e^{\frac{-2\pi ikj}{K}}$.

- The *ambiguity function* of u, $A : \mathbb{Z}_K \times \mathbb{Z}_K \to \mathbb{C}$ is defined as

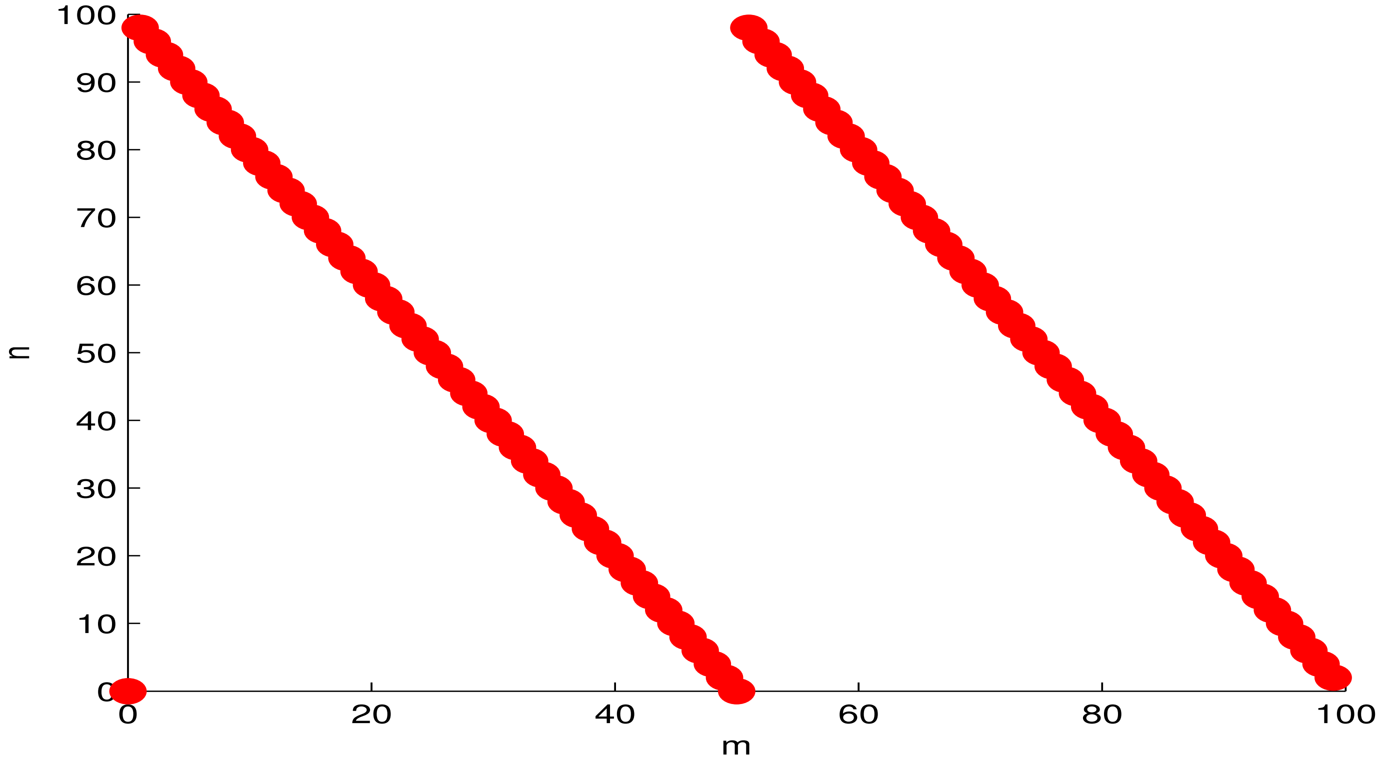
$$A_u[m,j] = C_{u,ue_j}[m] = \frac{1}{K}\sum_{k=0}^{K-1} u[k+m]\overline{u}[k]e^{\frac{2\pi ikj}{K}}.$$

- Analogue ambiguity function for $u \leftrightarrow U$, $\|u\|_2 = 1$, on $\mathbb{R}$:

$$A_u(t,\gamma) = \int_{\widehat{\mathbb{R}}} U(\omega - \frac{\gamma}{2})\overline{U(\omega + \frac{\gamma}{2})}e^{2\pi it(\omega+\frac{\gamma}{2})}d\omega$$

$$= \int u(s+t)\overline{u(s)}e^{2\pi is\gamma}ds.$$

Wiener CAZAC ambiguity, $K = 100, j = 2$



Rationale and theorem

Different CAZACs exhibit different behavior in their ambiguity plots, according to their construction method. Thus, the ambiguity function reveals localization properties of different constructions.

Theorem 1 *Given K odd, $\zeta = e^{\frac{2\pi i}{K}}$, and $u[k] = \zeta^{k^2}, \ k \in \mathbb{Z}_K$*

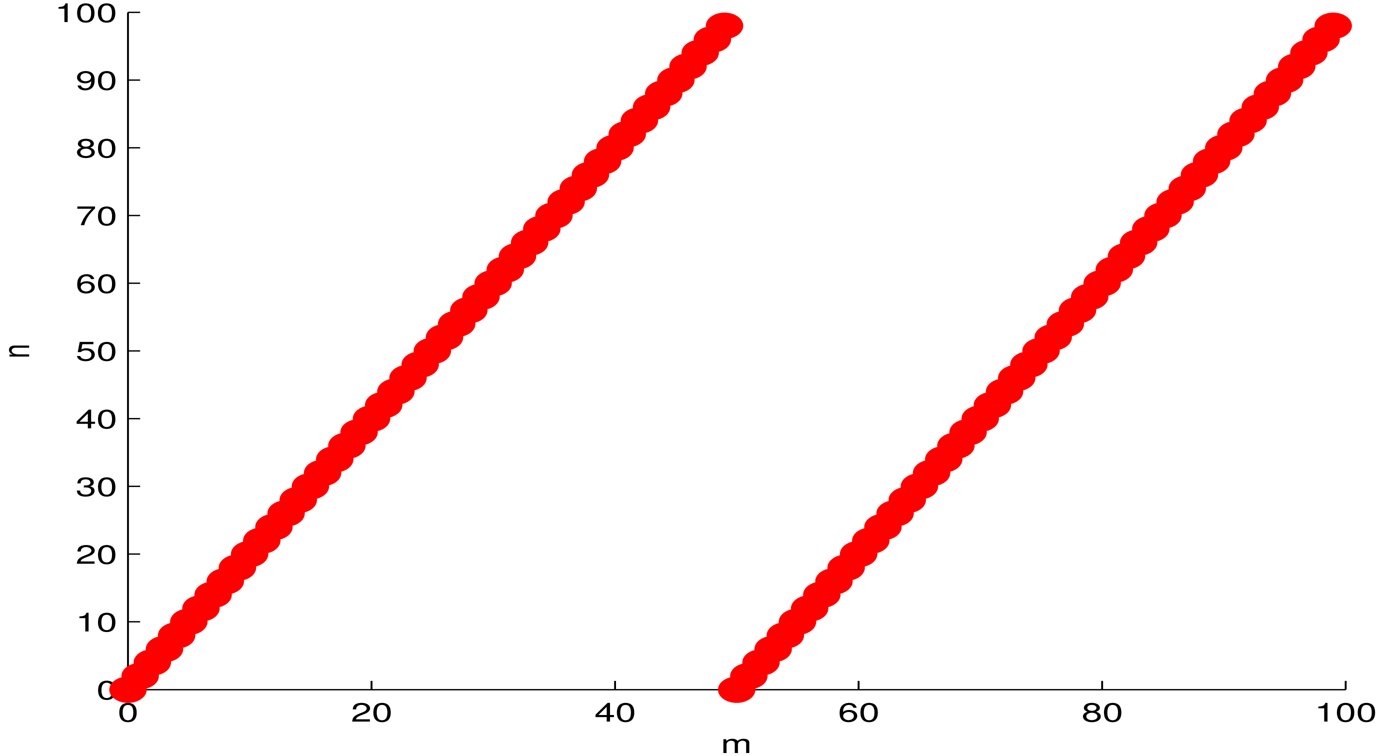
- $1 \leq k \leq K - 2$ *odd implies*

$$A[m,k] = e^{\pi i (K-k)^2/K} \text{ for } m = \frac{1}{2}(K-k), \text{ and } 0 \text{ elsewhere}$$

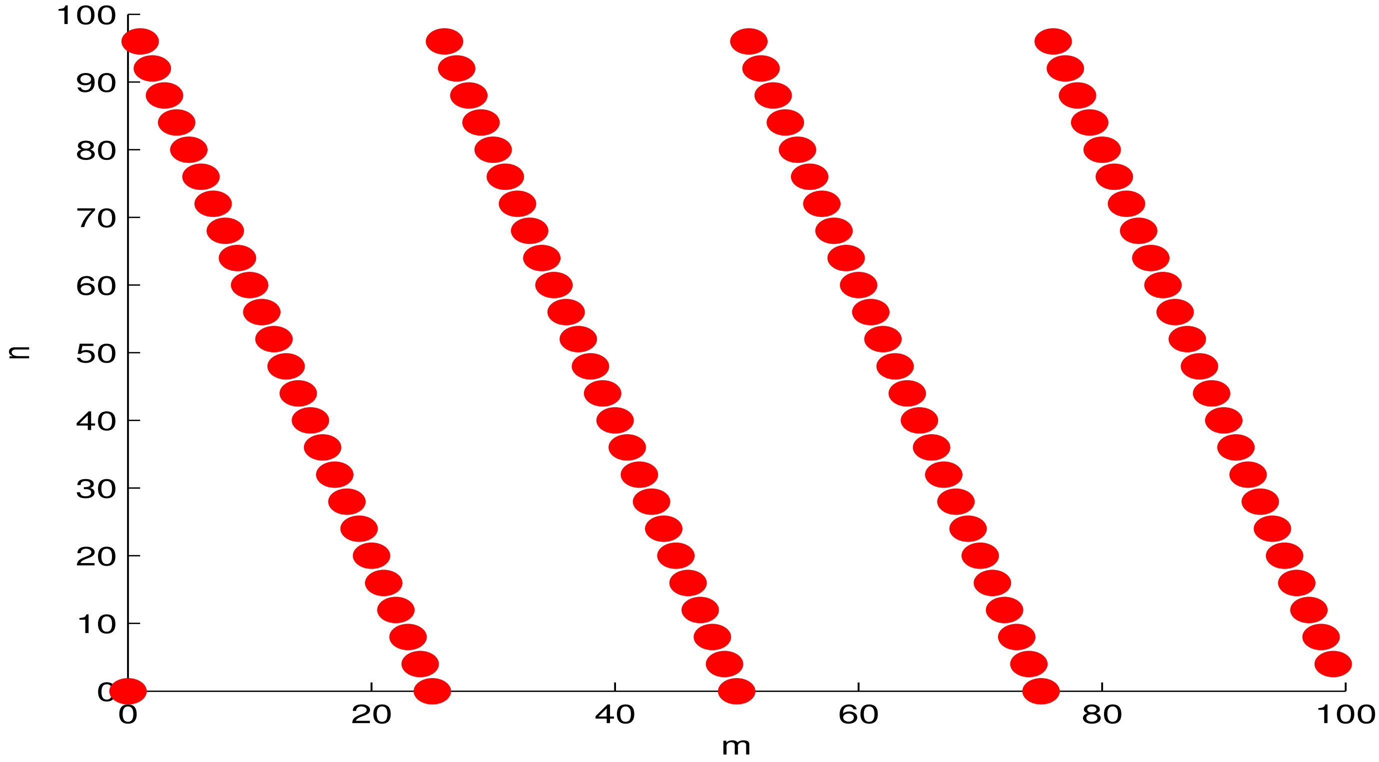
- $2 \leq k \leq K - 1$ *even implies*

$$A[m,k] = e^{\pi i (2K-k)^2/K} \text{ for } m = \frac{1}{2}(2K-k), \text{ and } 0 \text{ elsewhere}$$

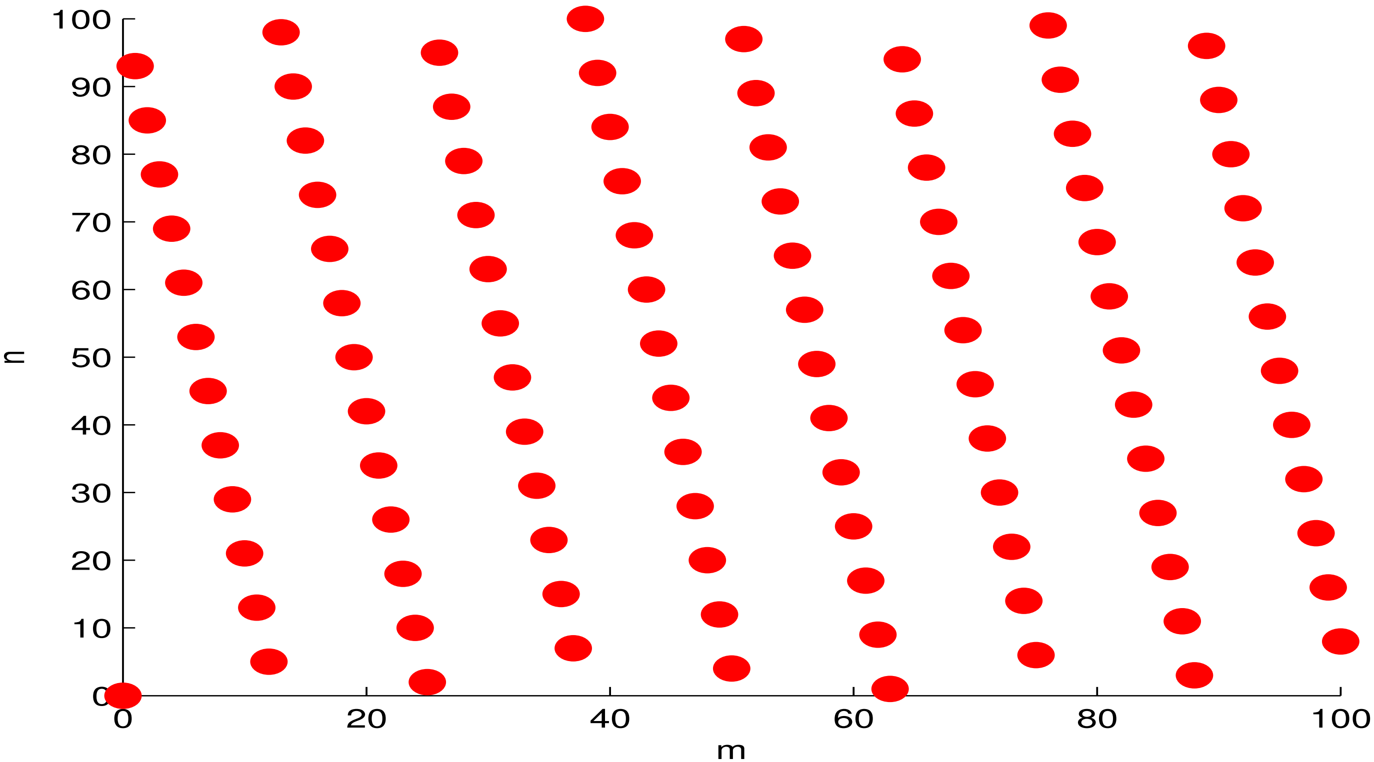
Wiener CAZAC ambiguity, $K = 100, j = 9$



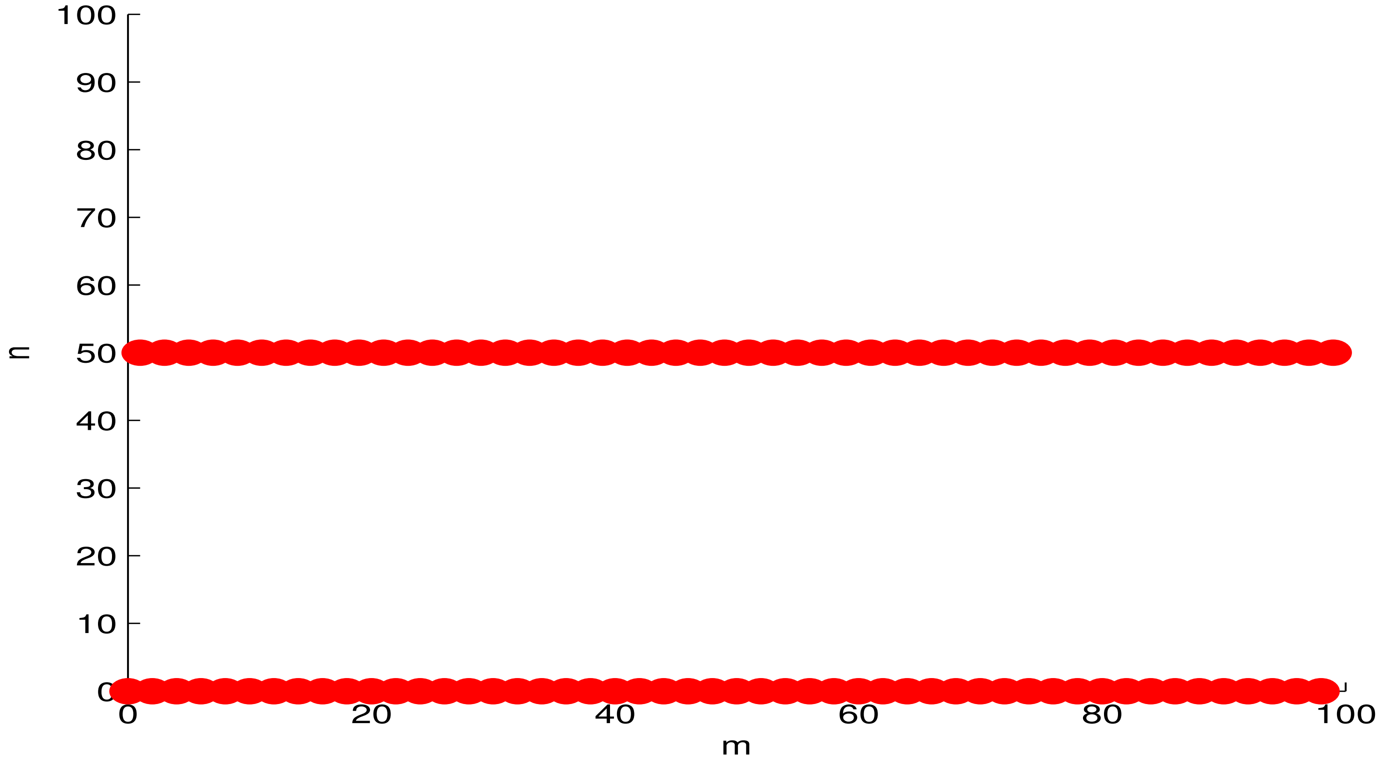
Wiener CAZAC ambiguity, $K = 100, j = 4$



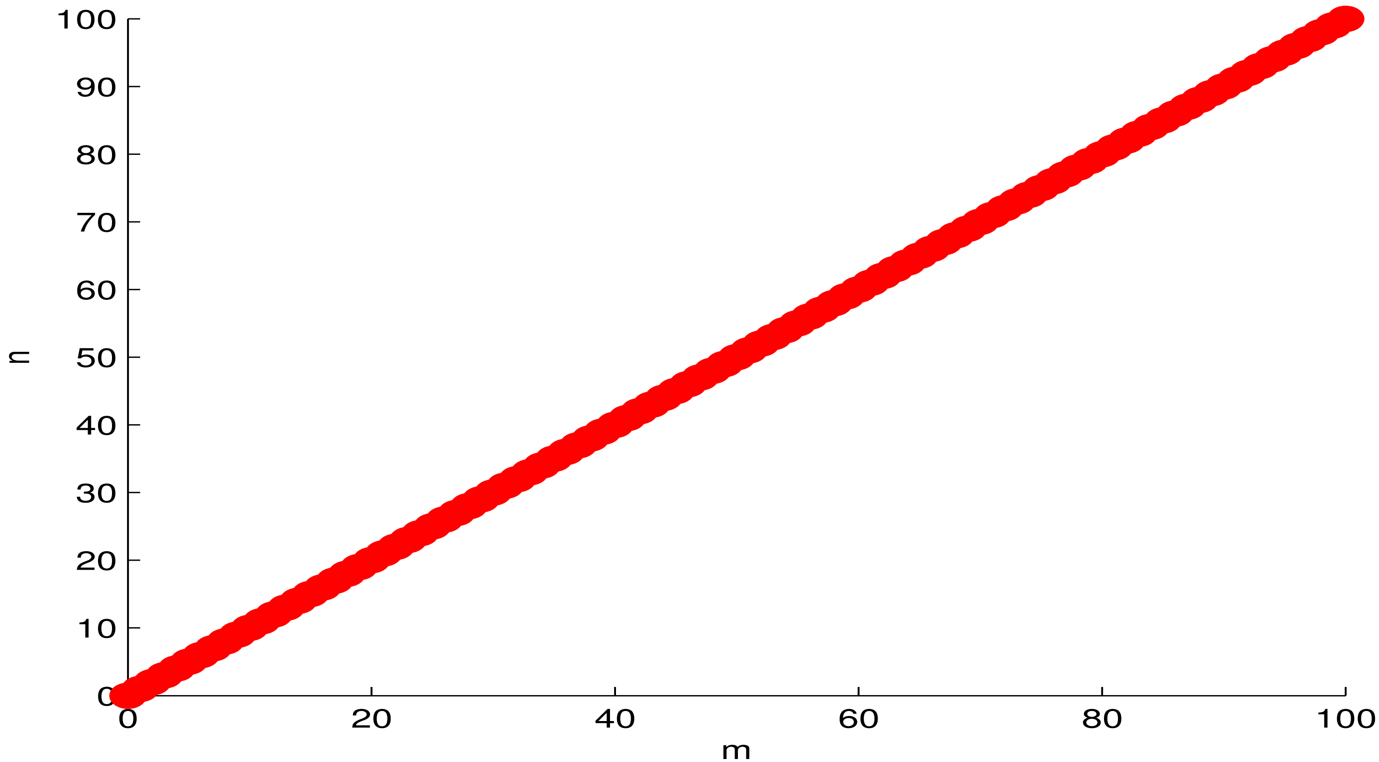
Wiener CAZAC ambiguity, $K = 101, j = 4$



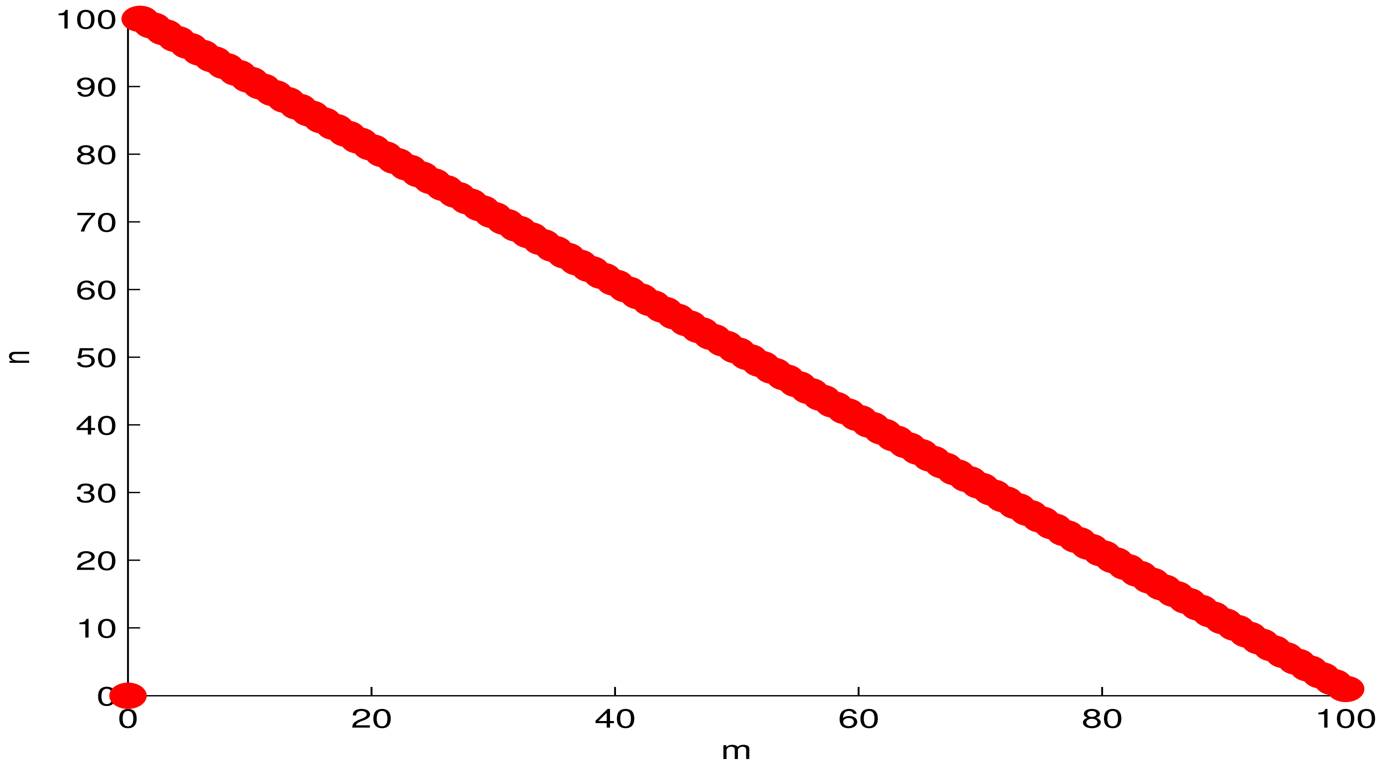
Wiener CAZAC ambiguity, $K = 100, j = 5$



Wiener CAZAC ambiguity, $K = 101, j = 5$



Wiener CAZAC ambiguity, $K = 101, j = 5$



Finite frames

Frames

Frames $F = \{e_n\}_{n=1}^N$ for d-dimensional Hilbert space H, e.g., $H = \mathbb{K}^d$, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$.

- Any spanning set of vectors in $\mathbb{K}^d$ is a *frame* for $\mathbb{K}^d$.
- $F \subseteq \mathbb{K}^d$ is A-*tight* if

$$\forall x \in \mathbb{K}^d,\ A\|x\|^2 = \sum_{n=1}^{N} |\langle x, e_n \rangle|^2$$

- If $\{e_n\}_{n=1}^N$ is a finite unit norm tight frame (FUN-TF) for $\mathbb{K}^d$, with frame constant A, then $A = N/d$.
- Let $\{e_n\}$ be an A-unit norm TF for any separable Hilbert space H. $A \geq 1$, and $A = 1 \Leftrightarrow \{e_n\}$ is an ONB for H (*Vitali*).

Properties and examples of FUN-TFs

- Frames give redundant signal representation to compensate for hardware errors, to ensure numerical stability, and to minimize the effects of noise.
- Thus, if certain types of noises are known to exist, then the FUN-TFs are constructed using this information.
- Orthonormal bases, vertices of Platonic solids, kissing numbers (sphere packing and error correcting codes) are FUN-TFs.
- The vector-valued CAZAC – FUN-TF problem: Characterize $u : \mathbb{Z}_K \longrightarrow \mathbb{C}^d$ which are CAZAC FUN-TFs.

Recent applications of FUN-TFs

- Robust transmission of data over erasure channels such as the internet [Casazza, Goyal, Kelner, Kovačević]
- Multiple antenna code design for wireless communications [Hochwald, Marzetta,T. Richardson, Sweldens, Urbanke]
- Multiple description coding [Goyal, Heath, Kovačević, Strohmer,Vetterli]
- Quantum detection [Bölcskei, Eldar, Forney, Oppenheim, Kebo, B]
- Grassmannian “min-max” waveforms [Calderbank, Conway, Sloane, et al., Kolesar, B]

DFT FUN-TFs

- $N \times d$ submatrices of the $N \times N$ DFT matrix are FUN-TFs for $\mathbb{C}^d$. These play a major role in finite frame $\Sigma\Delta$-quantization.

$$N = 8,\, d = 5 \qquad \frac{1}{\sqrt{5}} \begin{bmatrix} * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \end{bmatrix}$$

$$x_m = \frac{1}{5}(e^{2\pi i \frac{m}{8}}, e^{2\pi i m \frac{2}{8}}, e^{2\pi i m \frac{5}{8}}, e^{2\pi i m \frac{6}{8}}, e^{2\pi i m \frac{7}{8}})$$

$$m = 1, \ldots, 8.$$

- "Sigma-Delta" Super Audio CDs - but not all authorities are fans.

Characterization of FUN-TFs

For the Hilbert space $H = \mathbb{R}^d$ and N, consider $\{x_n\}_1^N \in S^{d-1} \times ... \times S^{d-1}$ and

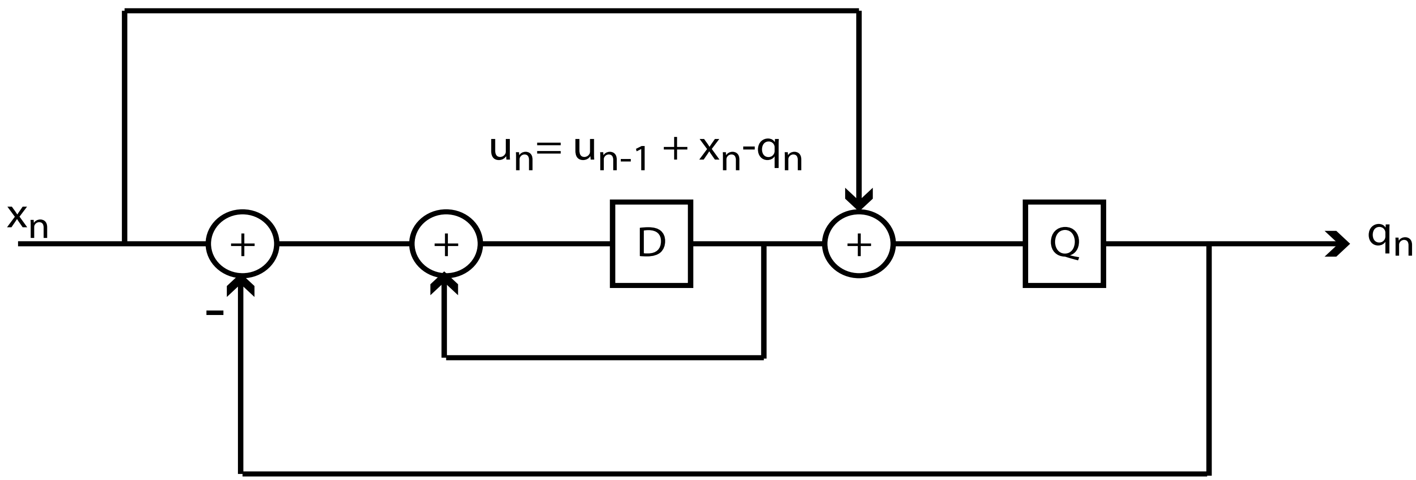
$$TFP(\{x_n\}) = \Sigma_{m=1}^N \Sigma_{n=1}^N | < x_m, x_n > |^2.$$

- **Theorem** Let $N \leq d$. The minimum value of TFP, for the frame force and N variables, is N; and the *minimizers* are precisely the orthonormal sets of N elements for $\mathbb{R}^d$.
- **Theorem** Let $N \geq d$. The minimum value of TFP, for the frame force and N variables, is N^2/d; and the *minimizers* are precisely the FUN-TFs of N elements for $\mathbb{R}^d$.
- **Problem** Find FUN-TFs analytically, effectively, computationally.

Sigma-Delta quantization – theory and implementation

Given u_0 and $\{x_n\}_{n=1}$

$u_n = u_{n-1} + x_n - q_n$
$q_n = Q(u_{n-1} + x_n)$



First Order ΣΔ

A quantization problem

Qualitative Problem Obtain *digital* representations for class X, suitable for storage, transmission, recovery.

Quantitative Problem Find dictionary $\{e_n\} \subseteq X$:

1. Sampling [continuous range $\mathbb{K}$ is not digital]

$$\forall x \in X, \;\; x = \sum x_n e_n, \;\; x_n \in \mathbb{K} \; (\mathbb{R} \text{ or } \mathbb{C}).$$

2. Quantization. Construct finite alphabet $\mathcal{A}$ and

$$Q : \; X \to \{\sum q_n e_n : \; q_n \in \mathcal{A} \subseteq \mathbb{K}\}$$

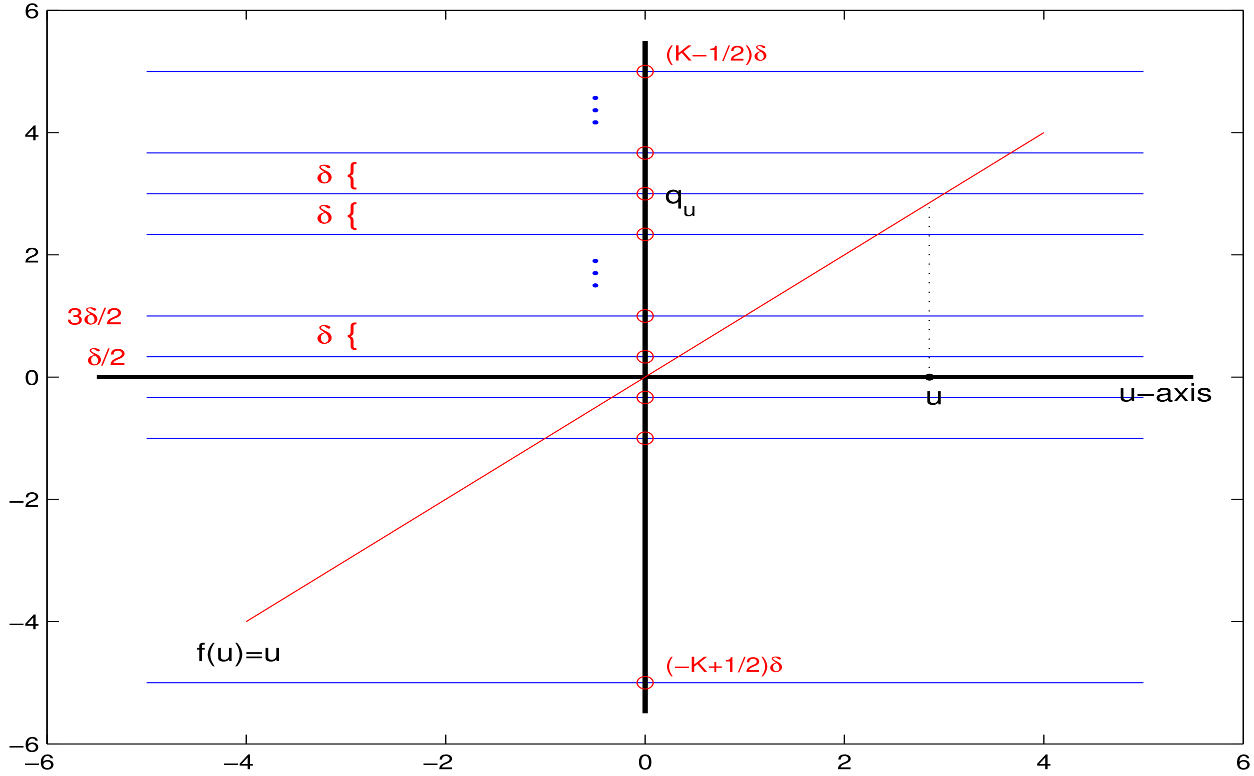
   such that $|x_n - q_n|$ and/or $\|x - Qx\|$ small.

Methods Fine quantization, e.g., PCM. Take $q_n \in \mathcal{A}$ close to given x_n. Reasonable in 16-bit (65,536 levels) digital audio.

Coarse quantization, e.g., $\Sigma\Delta$. Use fewer bits to exploit redundancy.

Quantization

$$\mathcal{A}_K^\delta = \{(-K+1/2)\delta, (-K+3/2)\delta, \ldots, (-1/2)\delta, (1/2)\delta, \ldots, (K-1/2)\delta\}$$



$$Q(u) = \arg\min\{|u-q| : q \in \mathcal{A}_K^\delta\} = q_u$$

PCM

Replace $x_n \leftrightarrow q_n = \arg\{\min|x_n - q| : q \in \mathcal{A}_K^\delta\}$. Then $\tilde{x} = \frac{d}{N}\sum_{n=1}^{N} q_n e_n$ satisfies

$$\|x - \tilde{x}\| \leq \frac{d}{N}\|\sum_{n=1}^{N}(x_n - q_n)e_n\| \leq \frac{d}{N}\frac{\delta}{2}\sum_{n=1}^{N}\|e_n\| = \frac{d}{2}\delta.$$

Not good!

Bennett's "white noise assumption"

Assume that $(\eta_n) = (x_n - q_n)$ is a sequence of independent, identically distributed random variables with mean 0 and variance $\frac{\delta^2}{12}$. Then the mean square error (MSE) satisfies

$$\text{MSE} = E\|x - \tilde{x}\|^2 \leq \frac{d}{12A}\,\delta^2 = \frac{(d\delta)^2}{12N}$$

$\mathcal{A}_1^2 = \{-1, 1\}$ and E_7

Let $x = (\frac{1}{3}, \frac{1}{2})$, $E_7 = \{(\cos(\frac{2n\pi}{7}), \sin(\frac{2n\pi}{7}))\}_{n=1}^{7}$. Consider quantizers with $\mathcal{A} = \{-1, 1\}$.

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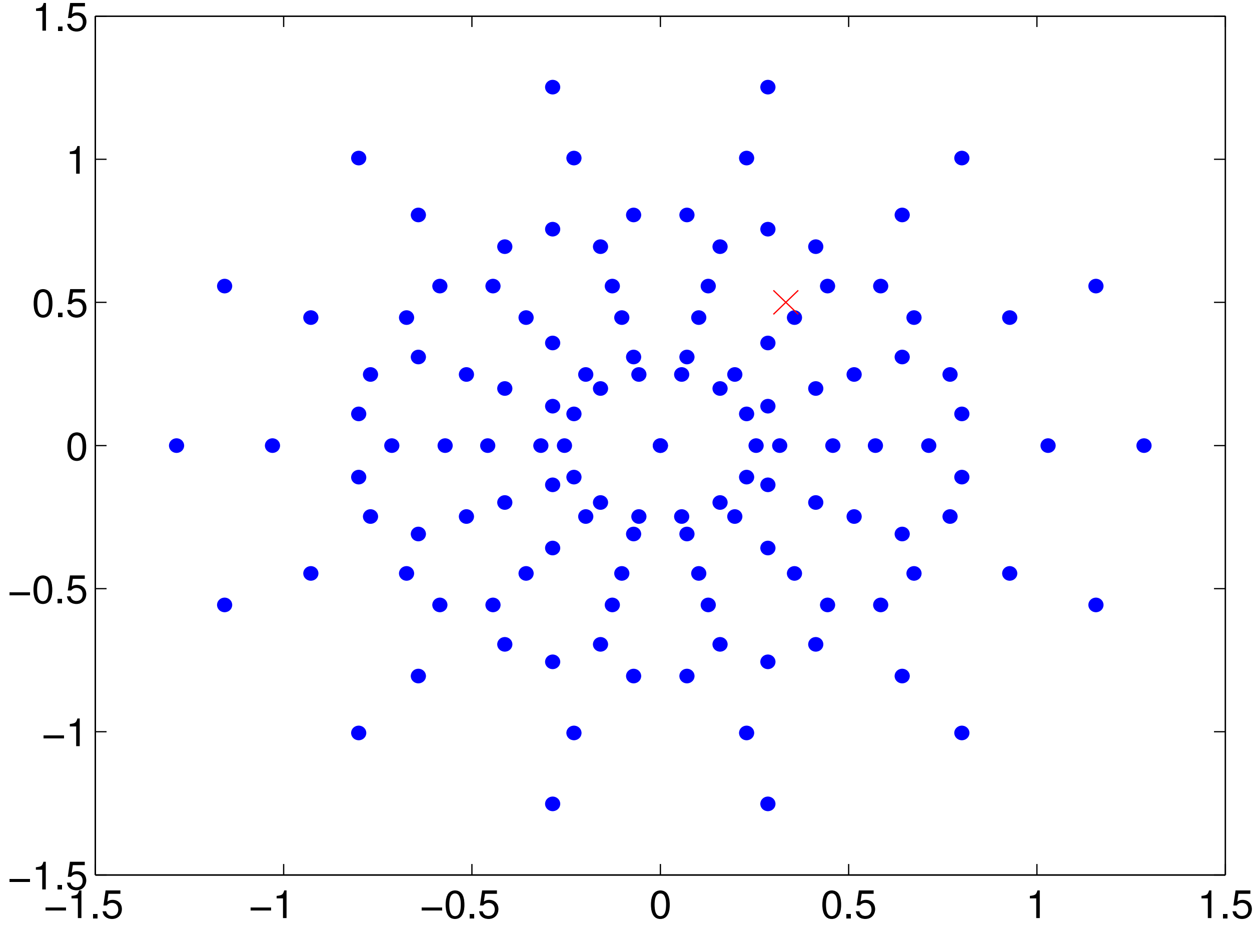

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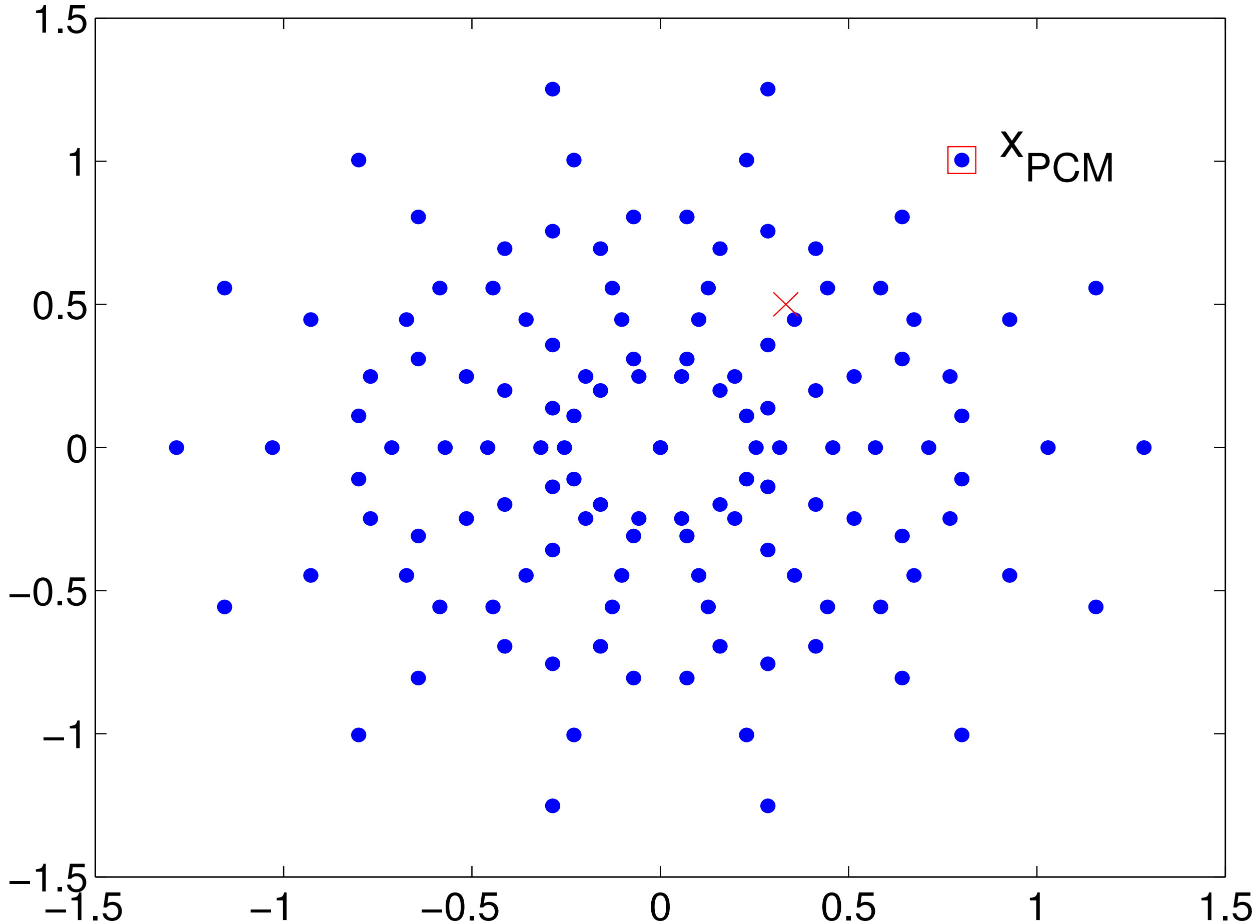

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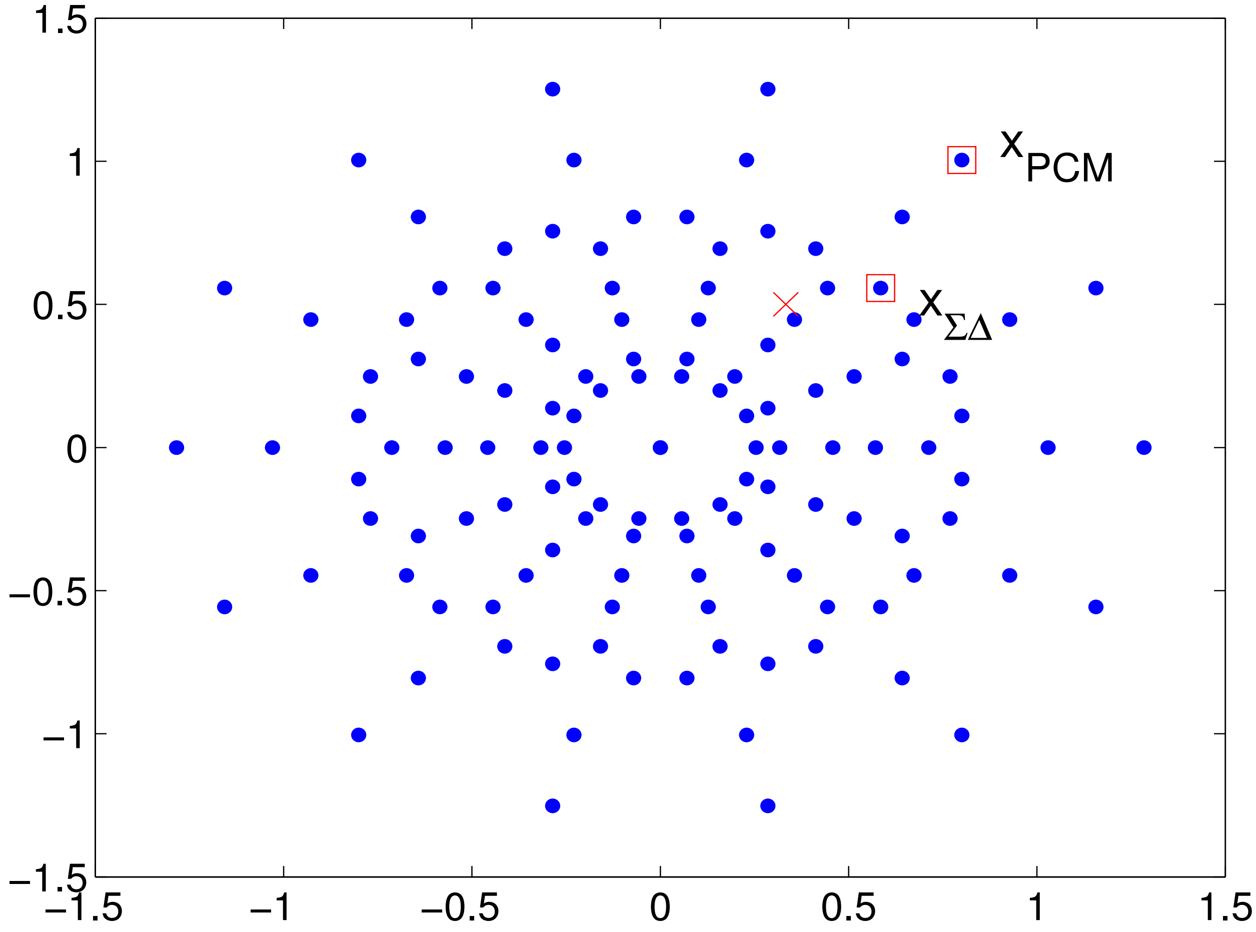


$\Sigma\Delta$ quantizers for finite frames

Let $F = \{e_n\}_{n=1}^N$ be a frame for $\mathbb{R}^d$, $x \in \mathbb{R}^d$.

Define $x_n = \langle x, e_n \rangle$.

Fix the ordering p, a permutation of $\{1, 2, \ldots, N\}$.

Quantizer alphabet $\mathcal{A}_K^\delta$

Quantizer function $Q(u) = \arg\{\min |u - q| : q \in \mathcal{A}_K^\delta\}$

Define the *first-order* $\Sigma\Delta$ *quantizer* with ordering p and with the quantizer alphabet $\mathcal{A}_K^\delta$ by means of the following recursion.

$$\begin{aligned} u_n - u_{n-1} &= x_{p(n)} - q_n \\ q_n &= Q(u_{n-1} + x_{p(n)}) \end{aligned}$$

where $u_0 = 0$ and $n = 1, 2, \ldots, N$.

Sigma-Delta quantization – background

- History from 1950s.
- Treatises of Candy, Temes (1992) and Norsworthy, Schreier, Temes (1997).
- PCM for finite frames and $\Sigma - \Delta$ for PW_Ω:
  Bölcskei, Daubechies, DeVore, Goyal, Güntürk, Kovačeviè, Thao, Vetterli.
- Combination of $\Sigma - \Delta$ and finite frames:
  Powell, Yılmaz, and B.
- Subsequent work based on this $\Sigma - \Delta$ finite frame theory:
  Bodman and Paulsen; Boufounos and Oppenheim; Jimenez and Yang Wang; Lammers, Powell, and Yılmaz.
- Genuinely apply it.

Stability

The following stability result is used to prove error estimates.

Proposition If the frame coefficients $\{x_n\}_{n=1}^N$ satisfy

$$|x_n| \leq (K - 1/2)\delta, \quad n = 1, \cdots, N,$$

then the state sequence $\{u_n\}_{n=0}^N$ generated by the first-order $\Sigma\Delta$ quantizer with alphabet $\mathcal{A}_K^\delta$ satisfies $|u_n| \leq \delta/2, n = 1, \cdots, N.$

- The first-order $\Sigma\Delta$ scheme is equivalent to

$$u_n = \sum_{j=1}^{n} x_{p(j)} - \sum_{j=1}^{n} q_j, \quad n = 1, \cdots, N.$$

- Stability results lead to tiling problems for higher order schemes.

Error estimate

- **Definition** Let $F = \{e_n\}_{n=1}^N$ be a frame for $\mathbb{R}^d$, and let p be a permutation of $\{1, 2, \ldots, N\}$. The *variation* $\sigma(F, p)$ is

$$\sigma(F, p) = \sum_{n=1}^{N-1} \|e_{p(n)} - e_{p(n+1)}\|.$$

- **Theorem** Let $F = \{e_n\}_{n=1}^N$ be an A-FUN-TF for $\mathbb{R}^d$. The approximation

$$\tilde{x} = \frac{d}{N} \sum_{n=1}^{N} q_n e_{p(n)}$$

generated by the first-order $\Sigma\Delta$ quantizer with ordering p and with the quantizer alphabet $\mathcal{A}_K^\delta$ satisfies

$$\|x - \tilde{x}\| \leq \frac{(\sigma(F, p) + 1)d}{N} \frac{\delta}{2}.$$

Harmonic frames

Zimmermann and Goyal, Kelner, Kovačević, Thao, Vetterli.

- $H = \mathbb{C}^d$. An *harmonic frame* $\{e_n\}_{n=1}^N$ for H is defined by the rows of the Bessel map L which is the complex N-DFT $N \times d$ matrix with $N - d$ columns removed.
- $H = \mathbb{R}^d$, d even. The harmonic frame $\{e_n\}_{n=1}^N$ is defined by the Bessel map L which is the $N \times d$ matrix whose nth row is

$$e_n^N = \sqrt{\frac{2}{d}} \left(\cos(\frac{2\pi n}{N}), \sin(\frac{2\pi n}{N}), \ldots, \cos(\frac{2\pi (d/2) n}{N}), \sin(\frac{2\pi (d/2) n}{N}) \right).$$

- Harmonic frames are FUN-TFs.
- Let E_N be the harmonic frame for $\mathbb{R}^d$ and let p_N be the identity permutation. Then

$$\forall N, \ \sigma(E_N, p_N) \leq \pi d(d+1).$$

Error estimate for harmonic frames

Theorem Let E_N be the harmonic frame for $\mathbb{R}^d$ with frame bound N/d. Consider $x \in \mathbb{R}^d$, $\|x\| \leq 1$, and suppose the approximation $\tilde{x}$ of x is generated by a first-order $\Sigma\Delta$ quantizer as before. Then

$$\|x - \tilde{x}\| \leq \frac{d^2(d+1)+d}{N} \frac{\delta}{2}.$$

- Hence, for harmonic frames (and all those with bounded variation),

$$\text{MSE}_{\Sigma\Delta} \leq \frac{C_d}{N^2} \delta^2.$$

- This bound is clearly superior asymptotically to

$$\text{MSE}_{\text{PCM}} = \frac{(d\delta)^2}{12N}.$$

$\Sigma\Delta$ and "optimal" PCM

The digital encoding

$$\text{MSE}_{\text{PCM}} = \frac{(d\delta)^2}{12N}$$

in PCM format leaves open the possibility that decoding (consistent nonlinear reconstruction, with additional numerical complexity this entails) could lead to

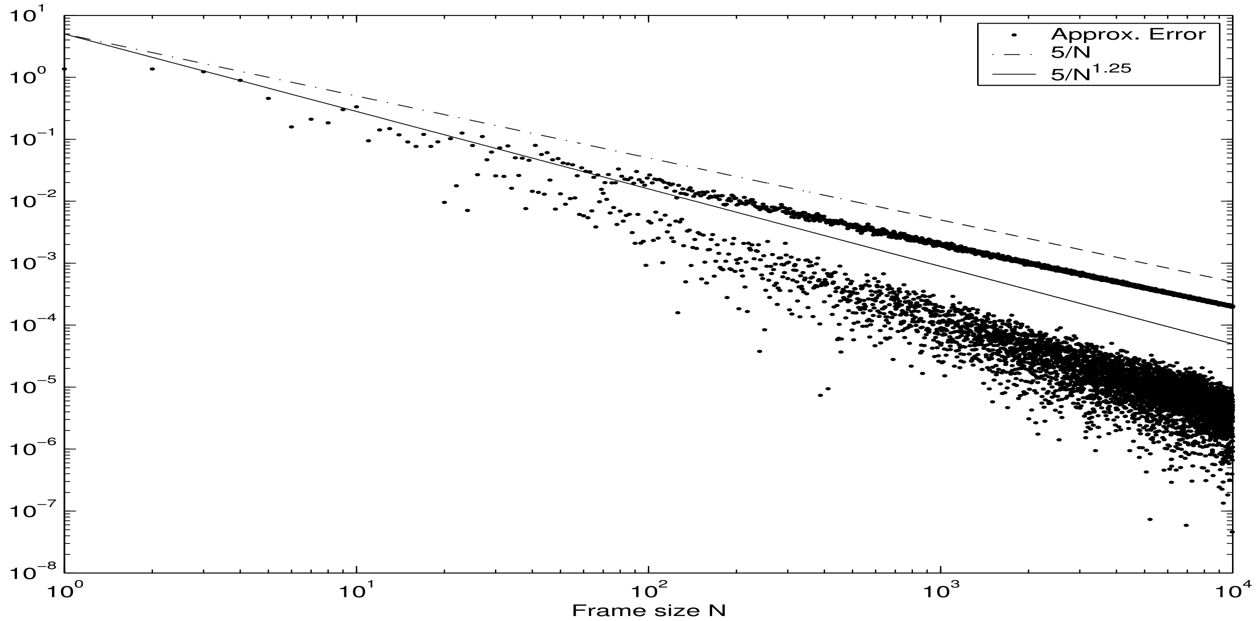
$$\text{“MSE}^{\text{opt}}_{\text{PCM}}\text{”} \ll O(\frac{1}{N}).$$

Goyal, Vetterli, Thao (1998) proved

$$\text{“MSE}^{\text{opt}}_{\text{PCM}}\text{”} \sim \frac{\tilde{C}_d}{N^2}\delta^2.$$

Theorem The first order $\Sigma\Delta$ scheme achieves the asymptotically optimal MSE_{PCM} for harmonic frames.

Even – odd



$E_N = \{e_n^N\}_{n=1}^N, e_n^N = (\cos(2\pi n/N), \sin(2\pi n/N))$. Let $x = (\frac{1}{\pi}, \sqrt{\frac{3}{17}})$.

$$x = \frac{d}{N} \sum_{n=1}^{N} x_n^N e_n^N, \quad x_n^N = \langle x, e_n^N \rangle.$$

Let $\widetilde{x}_N$ be the approximation given by the 1st order $\Sigma\Delta$ quantizer with alphabet $\{-1, 1\}$ and natural ordering. log-log plot of $||x - \widetilde{x}_N||$.

Improved estimates

$E_N = \{e_n^N\}_{n=1}^N$, Nth roots of unity FUN-TFs for $\mathbb{R}^2$, $x \in \mathbb{R}^2$, $||x|| \leq (K - 1/2)\delta$.

Quantize $$x = \frac{d}{N} \sum_{n=1}^{N} x_n^N e_n^N, \quad x_n^N = \langle x, e_n^N \rangle$$

using 1st order $\Sigma\Delta$ scheme with alphabet $\mathcal{A}_K^\delta$.

Theorem If N is even and large then $||x - \widetilde{x}|| \leq B_x \frac{\delta \log N}{N^{5/4}}$.

If N is odd and large then $A_x \frac{\delta}{N} \leq ||x - \widetilde{x}|| \leq B_x \frac{(2\pi+1)d}{N} \, \frac{\delta}{2}$.

- The proof uses a theorem of Güntürk (from complex or harmonic analysis); and Koksma and Erdös-Turán inequalities and van der Corput lemma (from analytic number theory).
- The Theorem is true for harmonic frames for $\mathbb{R}^d$.

Proof of Improved Estimates theorem

- If N is even and large then $||x - \widetilde{x}|| \leq B_x \frac{\delta \log N}{N^{5/4}}$.

  If N is odd and large then $A_x \frac{\delta}{N} \leq ||x - \widetilde{x}|| \leq B_x \frac{(2\pi+1)d}{N} \frac{\delta}{2}$.
- $\forall N, \{e_n^N\}_{n=1}^N$ is a FUN-TF.

$$x - \widetilde{x}_N = \frac{d}{N} \left(\sum_{n=1}^{N-2} v_n^N (f_n^N - f_{n+1}^N) + v_{N-1}^N f_{N-1}^N + u_N^N e_N^N \right)$$

$$f_n^N = e_n^N - e_{n+1}^N, \quad v_n^N = \sum_{j=1}^{n} u_j^N, \quad \widetilde{u}_n^N = \frac{u_n^N}{\delta}$$

- To bound v_n^N.

Koksma Inequality

- **Discrepancy**
  The discrepancy D_N of a finite sequence $x_1, \ldots, x_N$ of real numbers is
  $D_N = D_N(x_1, \ldots, x_N) = \sup_{0 \le \alpha < \beta \le 1} \left| \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{[\alpha,\beta)}(\{x_n\}) - (\beta - \alpha) \right|,$
  where $\{x\} = x - \lfloor x \rfloor$.

- **Koksma Inequality**
  $g : [-1/2, 1/2) \to \mathbb{R}$ of bounded variation and $\{\omega_j\}_{j=1}^{n} \subset [-1/2, 1/2) \Longrightarrow$

$$\left| \frac{1}{n} \sum_{j=1}^{n} g(\omega_j) - \int_{-\frac{1}{2}}^{\frac{1}{2}} g(t)dt \right| \le \mathrm{Var}(g)\mathrm{Disc}\Big(\{\omega_j\}_{j=1}^{n}\Big).$$

- With $g(t) = t$ and $\omega_j = \widetilde{u}_j^N$, $\boxed{|v_n^N| \le n\delta \mathrm{Disc}\Big(\{\widetilde{u}_j^N\}_{j=1}^{n}\Big).}$

Erdös-Turán Inequality

$$\exists C > 0, \forall K, \mathrm{Disc}\Big(\{\widetilde{u}_n^N\}_{n=1}^j\Big) \le C\left(\frac{1}{K} + \frac{1}{j}\sum_{k=1}^{K}\frac{1}{k}\Big|\sum_{n=1}^{j} e^{2\pi i k \widetilde{u}_n^N}\Big|\right).$$

To approximate the exponential sum.

Approximation of Exponential Sum

(1) Güntürk'sProposition

$\forall N, \exists X_N \in \mathcal{B}_{\Omega/N}$
such that $\forall n = 0, \ldots, N$,
$X_N(n) = u_n^N + c_n \frac{\delta}{2},\ c_n \in \mathbb{Z}$
and $\forall t, \left| X_N'(t) - h\left(\frac{t}{N}\right) \right| \leq B \frac{1}{N}$

(2) Bernstein's Inequality

If $x \in \mathcal{B}_\Omega$, then $\|x^{(r)}\|_\infty \leq \Omega^r \|x\|_\infty$

- $\widehat{\mathcal{B}}_\Omega = \{T \in A'(\widehat{\mathbb{R}})\ :\ \text{supp}T \subseteq [-\Omega, \Omega]\}$
- $\mathcal{M}_\Omega = \{h \in \mathcal{B}_\Omega\ :\ h' \in L^\infty(\mathbb{R})$ and all zeros of h' on $[0,1]$ are simple$\}$
- We assume $\exists h \in \mathcal{M}_\Omega$ such that $\forall N$ and $\forall\, 1 \leq n \leq N,\ h(n/N) = x_n^N$.

Approximation of Exponential Sum

(1) Güntürk’sProposition

$\forall N, \exists X_N \in \mathcal{B}_{\Omega/N}$
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(2) Bernstein’s Inequality

If $x \in \mathcal{B}_\Omega$, then $\|x^{(r)}\|_\infty \leq \Omega^r \|x\|_\infty$

(1)+(2)

$$\forall t, \left| X_N''(t) - \frac{1}{N} h'\left(\frac{t}{N}\right) \right| \leq B \frac{1}{N^2}$$

- $\widehat{\mathcal{B}}_\Omega = \{ T \in A'(\widehat{\mathbb{R}})\ :\ $ supp$T \subseteq [-\Omega, \Omega] \}$
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Van der Corput Lemma

- Let a, b be integers with $a < b$, and let f satisfy $f'' \geq \rho > 0$ on $[a, b]$ or $f'' \leq -\rho < 0$ on $[a, b]$. Then

$$\Big|\sum_{n=a}^{b} e^{2\pi i f(n)}\Big| \leq \Big(\big|f'(b) - f'(a)\big| + 2\Big)\Big(\frac{4}{\sqrt{\rho}} + 3\Big).$$

- $\forall 0 < \alpha < 1, \exists N_\alpha$ such that $\forall N \geq N_\alpha$,

$$\Big|\sum_{n=1}^{j} e^{2\pi i k \widetilde{u}_n^N}\Big| \leq B_x N^\alpha + B_x \frac{\sqrt{k} N^{1-\frac{\alpha}{2}}}{\sqrt{\delta}} + B_x \frac{k}{\delta}.$$

Choosing appropriate α and K

Putting $\alpha = 3/4,\ K = N^{1/4}$ yields

$$\exists \widetilde{N} \text{ such that } \forall N \geq \widetilde{N}, \mathrm{Disc}\left(\{\widetilde{u}_n^N\}_{n=1}^j\right) \leq B_x \frac{1}{N^{\frac{1}{4}}} + B_x \frac{N^{\frac{3}{4}} \log(N)}{j}$$

⇓

Conclusion

$$\boxed{\forall n = 1, \ldots, N,\ |v_n^N| \leq B_x \delta N^{\frac{3}{4}} \log N}$$

That's all folks!

Norbert Wiener Center