Worksheet for Section 7.2

1. Let $f(x) = e^{(x^2)}$ and $g(x) = (e^x)^2$.

   (a) Find $f'(x)$ and $g'(x)$.

   (b) One of the integrals $\int e^{(x^2)}\,dx$ and $\int (e^x)^2\,dx$ can be evaluated by the methods of Chapter 5. Determine which integral that is, and evaluate the integral.

2. Iodine 131, which has been used for treating cancer of the thyroid gland, is also used in order to detect leaks in water pipes. It has a half-life of (approximately) 8.14 days. Suppose that at noon today you have a bottle with 5 grams of Iodine 131.

   (a) Find a formula for the amount $f(t)$ of Iodine 131 in the bottle t hours after noon today (i.e., for each $t \geq 0$). (Hint: Section 4.4 might be helpful.)

   (b) Show that f has an inverse f^{-1}, and find a formula for f^{-1}.

   (c) In a complete sentence, indicate what the function f^{-1} tells us physically.

3. (a) Let b be any real number. Use the product rule for derivatives (and *not* the Law of Exponents) to show that $\dfrac{d}{dx}(e^{-x}e^{b+x}) = 0$. Consequently by Theorem 4.6, $e^{-x}e^{b+x}$ is a constant function.

   (b) By using (a) and letting $x = 0$, show that the constant function in (a) is e^b, so that $e^{-x}e^{b+x} = e^b$ for all x.

   (c) Use (b) with $b = 0$ to prove that $e^{-x} = 1/e^x$ for all real x.

   (d) Use (b) and (c) to prove that $e^{b+c} = e^b e^c$ for all b and c. (You have just proved the Law of Exponents!)

4. Let $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$, where μ and σ are constants and $\sigma > 0$.

   (a) By taking the derivative of f, show that the maximum value of f is $f(\mu) = \dfrac{1}{\sigma\sqrt{2\pi}}$. Then tell how you can know without taking the derivative of f that the maximum value of f occurs for $x = \mu$.

   (b) Show that $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$.

   (c) Let $\sigma = 1$ and $\mu = 0$. Use the left sum formula with $n = 50$ to get an approximate value of $\displaystyle\int_{-3}^{3} f(x)\,dx$. Guess what the limit of the values $\displaystyle\int_{-n}^{n} f(x)\,dx$ is when n approaches ∞.

Comments/Solutions for Worksheet Section 7.2

1. (a) $f'(x) = (2x)e^{(x^2)}$ and $g'(x) = 2e^x e^x = 2e^{2x}$.

   (b) The second can be integrated: $\displaystyle\int (e^x)^2\,dx = \int (e^{2x}\,dx = \frac{1}{2}e^{2x} + C$

2. Iodine 131, which has been used for treating cancer of the thyroid gland, is also used in order to detect leaks in water pipes. It has a half-life of (approximately) 8.14 days. Suppose that at noon today you have a bottle with 5 grams of Iodine 131.

   (a) $f(t) = 5e^{kt}$ for $t \geq 0$. Now $k = -(\ln 2)/8.14$, so $f(t) = 5e^{-t(\ln 2)/8.14}$.

   (b) Note that $f'(t) < 0$ for $t > 0$, so f^{-1} exists. To find the inverse, solve $f(t) = 5e^{-t(\ln 2)/8.14}$ for t.

   (c) f^{-1} tells us physically that for a given amount less than 5 grams, low long after noon it has been.

3. (a) Let b be any real number. By the product rule, $\dfrac{d}{dx}(e^{-x}e^{b+x}) = -e^{-x}e^{b+x} + e^{-x}e^{b+x} = 0$ for all x. Thus $e^{-x}e^{b+x} = C$ for some constant C.

   (b) By (a) with $x = 0$, we have $e^0 e^b = C$, so $e^b = C$. Thus $e^{-x}e^{b+x} = e^b$ for all x.

   (c) Let $b = 0$ in (b). Then $e^{-x}e^x = e^0 = 1$, so $e^{-x} = 1/e^x$ for all real x.

   (d) Let $x = c$. Then $e^{-c}e^{b+c} = e^b$, so $e^{b+c} = (e^b)/(e^{-c}) = e^b e^c$.

4. Let $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$, where μ and σ are constants and $\sigma > 0$.

   (a) By taking the derivative of f, show that the maximum value of f is $f(\mu) = \dfrac{1}{\sigma\sqrt{2\pi}}$. Then tell how you can know without taking the derivative of f that the maximum value of f occurs for $x = \mu$.

   (b) Show that $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$.

   (c) Let $\sigma = 1$ and $\mu = 0$. Use the left sum formula with $n = 50$ to get an approximate value of $\displaystyle\int_{-3}^{3} f(x)\,dx$. Guess what the limit of the values $\displaystyle\int_{-n}^{n} f(x)\,dx$ is when n approaches ∞.

5. Number 4 to come.