Worksheet for Section 9.3

1. (a) Suppose that $\lim_{n\to\infty} a_n = L$. Tell in a complete sentence why $\lim_{n\to\infty} a_{n+1} = L$.

   (b) Suppose that $\lim_{n\to\infty} a_n = 3$. Is it necessarily true that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$? Explain why it is true, or why it is not necessarily true. (Hint: Does (4) in Section 9.3 apply?)

   (c) Suppose that $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = \pi$. Prove that $\lim_{n\to\infty} a_n b_n = 0$.

2. When a superball is dropped onto a hardwood floor, it bounces up to approximately 80% of its original height. Suppose that the ball is dropped initially from a height of 5 feet above the floor, and let b_n = the maximum height of the nth bounce.

   (a) Evaluate b_1, b_2, and b_3. (b) Prove that $\lim_{n\to\infty} b_n = 0$.

3. Let the sequence $\{a_n\}_{n=1}^{\infty}$ be defined recursively by

$$a_1 = 1, \quad a_2 = \sin 1, \quad a_3 = \sin(\sin 1), \quad \text{and in general,} \quad a_{n+1} = \sin a_n, \text{ for all } n \geq 1$$

   (a) By Exercise 4.3.60(b) we know that $0 < a_{n+1} < a_n < 1$ for $n \geq 1$. From these inequalities we can use a theorem in Section 9.3 to conclude that $\{a_n\}_{n=1}^{\infty}$ has a limit. Write out the statement of the theorem.

   (b) Let the limit in part (a) be denoted L. Convince yourself and your group that

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sin a_n = \sin(\lim_{n\to\infty} a_n) = \sin L$$

   (c) Find the numerical value of L.

4. (a) Show that $\ln(n+1) - \ln n = \int_1^{n+1} \frac{1}{t}\,dt - \int_1^{n} \frac{1}{t}\,dt = \int_n^{n+1} \frac{1}{t}\,dt \geq \frac{1}{n+1}$.

   (b) Let $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n$, for $n = 1, 2, \ldots$. Use (a) to show that $\{a_n\}_{n=1}^{\infty}$ is a decreasing sequence. (Hint: Show that $a_n - a_{n+1} \geq 0$.)

   (c) Using the left sum of $\int_1^{n+1} \frac{1}{t}\,dt$ with partition $\{1, 2, ..., n+1\}$, show that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \geq \ln(n+1)$.

   (d) Use (c) and the definition of a_n to show that $a_n \geq 0$ for all n.

   (e) Use (b) and (d) to show that $\{a_n\}_{n=1}^{\infty}$ converges to a number r. ($r \approx 0.577216$, and is known as the Euler-Mascheroni constant).