CHAPTER 15

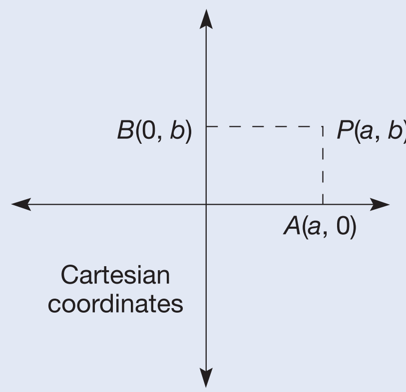
Geometry Using Coordinates

FOCUS ON René Descartes and Coordinate Geometry

The subjects of algebra and geometry had evolved on parallel tracks until René Descartes (1596–1650) developed a method of joining them.

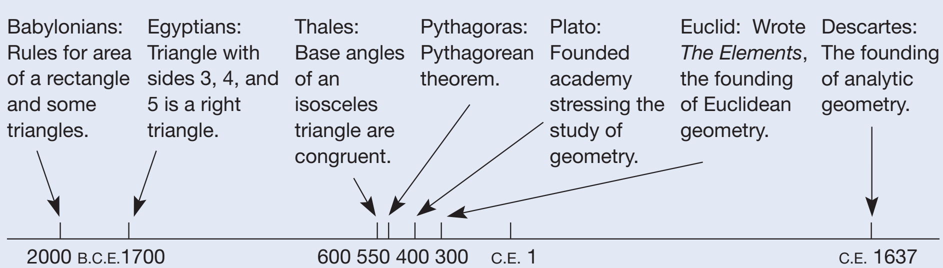
This important contribution made possible the development of the calculus. Because of this contribution, Descartes has been called the “father of modern mathematics.” The coordinate system used in analytic geometry is called the Cartesian coordinate system in his honor.

Descartes’ analytic geometry was designed to study the mathematical attributes of lines and curves by representing them via equations. These equations were obtained by picturing the curves in the plane. Each point, P, in the plane was labeled using pairs of numbers, or “coordinates,” determined by two perpendicular reference lines.



What is impressive about the coordinate approach is that geometric problems can be represented using algebraic equations. Then algebra can be applied to these equations without regard to their geometric representations. Finally, the result of the algebra can be reinterpreted to produce a solution to the original geometric problem.

Brief Timeline for Geometry



Problem-Solving Strategies

1. Guess and Test
2. Draw a Picture
3. Use a Variable
4. Look for a Pattern
5. Make a List
6. Solve a Simpler Problem
7. Draw a Diagram
8. Use Direct Reasoning
9. Use Indirect Reasoning
10. Use Properties of Numbers
11. Solve an Equivalent Problem
12. Work Backward
13. Use Cases
14. Solve an Equation
15. Look for a Formula
16. Do a Simulation
17. Use a Model
18. Use Dimensional Analysis
19. Identify Subgoals
20. **Use Coordinates**

STRATEGY 20

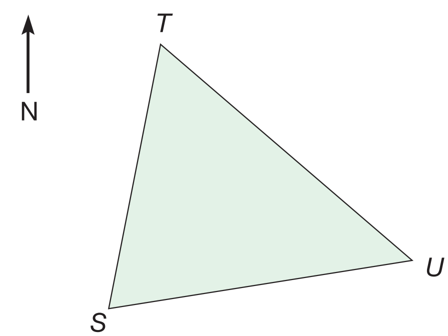
Use Coordinates

In many two-dimensional geometry problems, we can use a "grid" of squares overlaid on the plane, called a coordinate system, to gain additional information. This numerical information can then be used to solve problems about two-dimensional figures. Coordinate systems also can be used in three-dimensional space and on curved surfaces such as a sphere (e.g., the Earth).

INITIAL PROBLEM

A surveyor plotted a triangular building lot shown in the figure below. He described the locations of stakes T and U relative to stake S. For example, U is recorded as East 207′, North 35′. This would mean that to find stake U, one would walk due east 207 feet and then due north for 35 feet. From the perspective of the diagram shown, one would go right from point S 207 feet and up 35 feet to get to U. Use the information provided to find the area of the lot in square feet.

STAKE	POSITION RELATIVE TO S
U	East 207′, North 35′
T	East 40′, North 185′



CLUES

The Use Coordinates strategy may be appropriate when

- A problem can be represented using two variables.
- A geometry problem cannot easily be solved by using traditional Euclidean methods.
- A problem involves finding representations of lines or conic sections.
- A problem involves slope, parallel lines, perpendicular lines, and so on.
- The location of a geometric shape with respect to other shapes is important.
- A problem involves maps.

A solution of this Initial Problem is on page 844.

INTRODUCTION

Reflection from Research
An understanding of coordinate graphing develops over an extended period of time (Demana & Waits, 1988).

In this chapter we study geometry using the coordinate plane. Using a coordinate system on the plane, which was introduced in Section 9.3, we are able to derive many elegant geometrical results about lines, polygons, circles, and so on. In Section 15.1 we introduce the basic ideas needed to study geometry in the coordinate plane. Then in Section 15.2 we prove properties of geometric shapes using these concepts. Finally, Section 15.3 contains many interesting problems that can be solved using coordinate geometry.

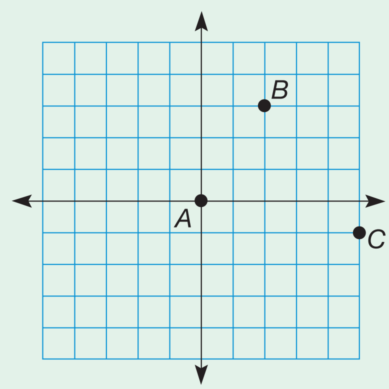
Key Concepts from NCTM Curriculum Focal Points

- **GRADE 3:** Describing and analyzing properties of two-dimensional shapes.
- **GRADE 6:** Writing, interpreting, and using mathematical expressions and equations.
- **GRADE 8:** Analyzing two- and three-dimensional space and figures by using distance and angle.

15.1 DISTANCE AND SLOPE IN THE COORDINATE PLANE

STARTING POINT

The points $A = (0, 0)$, $B = (2, 3)$, and $C = (5, -1)$ are plotted on the axes at the right. Locate a fourth point D such that A, B, C, and D are the vertices of a parallelogram. Justify your selection. Are there other points that will work? If so, how many?



Children's Literature
www.wiley.com/college/musser
See "The Fly on the Ceiling" by Julie Glass.

Distance

The use of coordinates as developed in Section 9.3 allows us to analyze many properties of geometric figures. For example, we can find distances between points in the plane using coordinates. Consider points $P(x_1, y_1)$ and $Q(x_2, y_2)$ (Figure 15.1). We can use point $R(x_2, y_1)$ to form a right triangle, $\triangle PQR$. Notice that the length of the horizontal segment $\overline{PR}$ is $x_2 - x_1$ and that the length of the vertical segment $\overline{QR}$ is $y_2 - y_1$. We wish to find the length of $\overline{PQ}$. By the Pythagorean theorem,

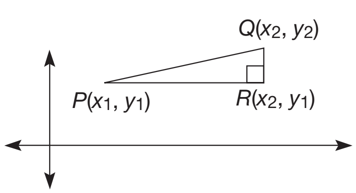


Figure 15.1

$$PQ^2 = PR^2 + QR^2,$$

so

$$PQ = \sqrt{PR^2 + QR^2}.$$

But

$$PR^2 = (x_2 - x_1)^2 \text{ and } QR^2 = (y_2 - y_1)^2.$$

Hence $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. This yields the following distance formula.

NCTM Standard
All students should find the distance between points along horizontal and vertical lines of a coordinate system.

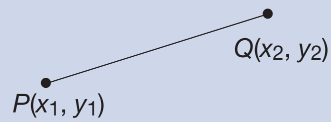
Connection to Algebra
The distance formula is an example of using variables to *efficiently* describe a method of computation.

THEOREM

Coordinate Distance Formula

If P is the point (x_1, y_1) and Q is the point (x_2, y_2), the distance from P to Q is

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$



We have verified the coordinate distance formula in the case that $x_2 \geq x_1$ and $y_2 \geq y_1$. However, this formula holds for *all* pairs of points in the plane. For example, if $x_1 \geq x_2$, we would use $(x_1 - x_2)^2$ in the formula. But $(x_1 - x_2)^2 = (x_2 - x_1)^2$, so that the formula will yield the same result.

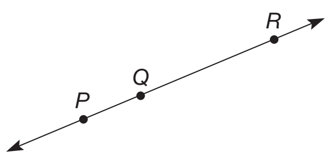


Figure 15.2

Collinearity Test We can use the coordinate distance formula to determine whether three points are collinear. Recall that points P, Q, and R are collinear with Q between P and R if and only if $PQ + QR = PR$; that is, the distance from P to R is the sum of the distances from P to Q and Q to R (Figure 15.2).

Example 15.1 Use the coordinate distance formula to show that the points $P = (-5, -4)$, $Q = (-2, -2)$, and $R = (4, 2)$ are collinear.

SOLUTION

$$PQ = \sqrt{[(-5) - (-2)]^2 + [(-4) - (-2)]^2} = \sqrt{9 + 4} = \sqrt{13}$$

$$QR = \sqrt{[(-2) - 4]^2 + [(-2) - 2]^2} = \sqrt{36 + 16} = 2\sqrt{13}$$

$$PR = \sqrt{[(-5) - 4]^2 + [(-4) - 2]^2} = \sqrt{81 + 36} = 3\sqrt{13}$$

Thus $PQ + QR = \sqrt{13} + 2\sqrt{13} = 3\sqrt{13} = PR$, so P, Q, and R are collinear. ■

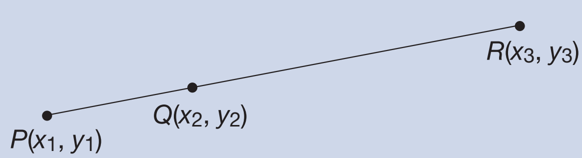
Example 15.1 is a special case of the following theorem.

THEOREM

Collinearity Test

Points $P(x_1, y_1)$, $Q(x_2, y_2)$, and $R(x_3, y_3)$ are collinear with Q between P and R if and only if $PQ + QR = PR$, or, equivalently,

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} + \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2} = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}.$$



Midpoint Formula Next we determine the midpoint of a line segment.

Consider $P(x_1, y_1)$ and $Q(x_2, y_2)$. Let $R(x_2, y_1)$ be the vertex of a right triangle that has $\overline{PQ}$ as the hypotenuse [see Figure 15.3(a)]. Then $\left(\frac{x_1 + x_2}{2}, y_1\right)$ and $\left(x_2, \frac{y_1 + y_2}{2}\right)$ are midpoints of $\overline{PR}$ and $\overline{QR}$, respectively [see Figure 15.3(a)]. Now let M be the intersection of the vertical line through midpoint $\left(\frac{x_1 + x_2}{2}, y_1\right)$ and the horizontal line through midpoint $\left(x_2, \frac{y_1 + y_2}{2}\right)$ [see Figure 15.3(b)]. Hence the coordinates of the midpoint M are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ We can verify that M is the midpoint of segment $\overline{PQ}$ by showing that $PM = MQ$ and that P, M, and Q are collinear [see Figure 15.3(b)]. This is left for the Problem Set.

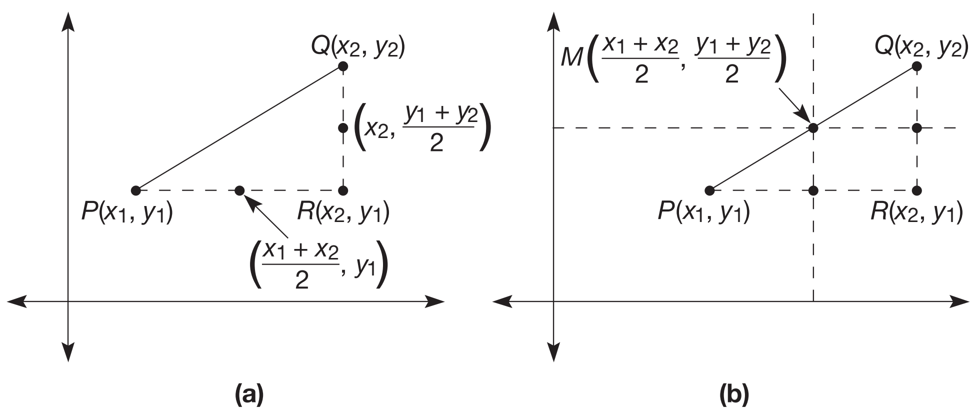


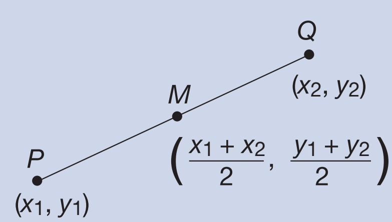
Figure 15.3

THEOREM

Midpoint Formula

If P is the point (x_1, y_1) and Q is the point (x_2, y_2), the midpoint, M, of $\overline{PQ}$ is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$



In words, the coordinates of the midpoint of a segment are the averages of the x-coordinates and y-coordinates of the endpoints, respectively.

Example 15.2 Find the coordinates of the midpoints of the three sides of $\triangle ABC$, where $A = (-6, 0)$, $B = (0, 8)$, $C = (10, 0)$ [Figure 15.4(a)].

SOLUTION

$$\text{midpointo f } \overline{AB} = \left(\frac{-6+0}{2}, \frac{0+8}{2}\right) = (-3, 4)$$

$$\text{midpointo f } \overline{BC} = \left(\frac{0+10}{2}, \frac{8+0}{2}\right) = (5, 4)$$

$$\text{midpointo f } \overline{AC} = \left(\frac{-6+10}{2}, \frac{0+0}{2}\right) = (2, 0) \text{ [Figure 15.4(b)]}$$

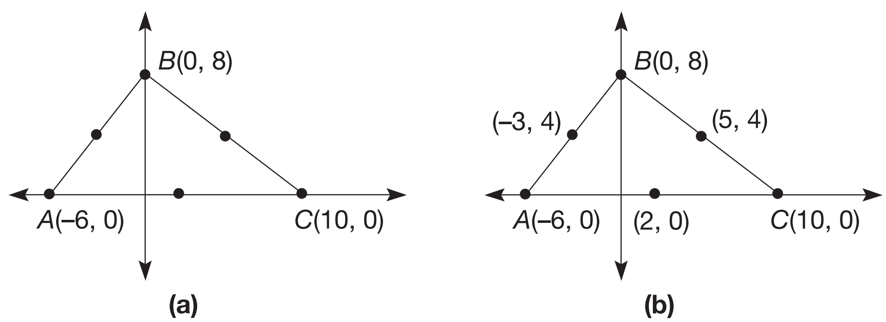


Figure 15.4

Slope

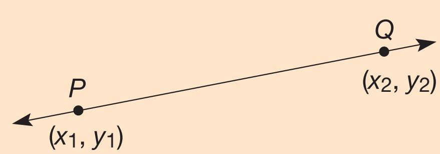
The slope of a line is a measure of its inclination from the horizontal.

NCTM Standard
All students should explore relationships between symbolic expressions and graphs of lines, paying particular attention to the meaning of intercept and slope.

DEFINITION

Slope of a Line

Suppose that P is the point (x_1, y_1) and Q is the point (x_2, y_2).



The **slope of line** $\overleftrightarrow{PQ}$ is the ratio $\frac{y_2 - y_1}{x_2 - x_1}$, provided $x_1 \neq x_2$. If $x_1 = x_2$, that is, $\overleftrightarrow{PQ}$ is vertical, then the slope of $\overleftrightarrow{PQ}$ is undefined.

Note that $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$, so that it does not matter which endpoint we use first in computing the slope of a line. However, we must be consistent in the numerator and denominator; that is, we subtract the coordinates of Q from the coordinates of P, or vice versa. The slope of a line is often defined informally as "rise over the run." The **slope of a line segment** is the slope of the line containing it.

Example 15.3 Find the slope of these lines: (a) line l containing $P(-8, -6)$ and $Q(10, 5)$, (b) line m containing $R(2, 1)$ and $S(20, 12)$, and (c) line n containing $T(-3, 11)$ and $U(4, 11)$ (Figure 15.5).

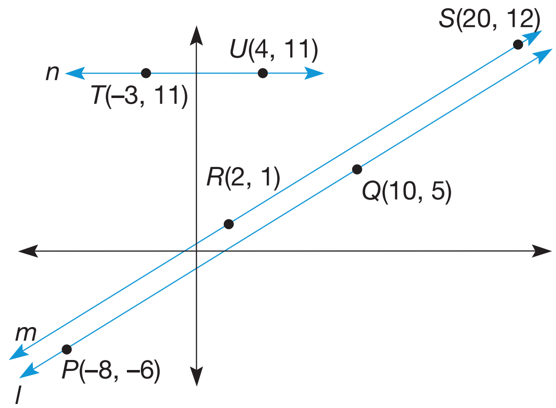


Figure 15.5

SOLUTION

a. Using the coordinates of P and Q, the slope of $l = \dfrac{5-(-6)}{10-(-8)} = \dfrac{11}{18}$.

b. Using the coordinates of R and S, the slope of $m = \dfrac{12-1}{20-2} = \dfrac{11}{18}$. Hence lines l and m have equal slopes.

c. Using points T and U, the slope of line $n = \dfrac{11-11}{4-(-3)} = 0$. Note that line n is horizontal. ■

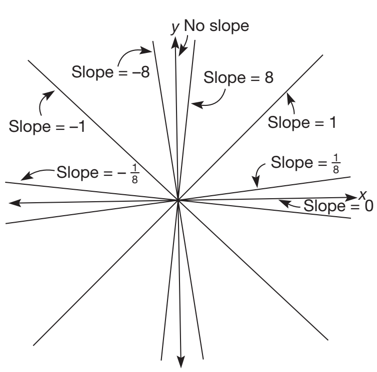


Figure 15.6

Figure 15.6 shows examples of lines with various slopes. A horizontal line, such as line n in Example 15.3, has slope 0. As the slope increases, the line "rises" to the right. A line that rises steeply from left to right has a large positive slope. A vertical line has no slope. On the other hand, a line that declines steeply from left to right has a *small* negative slope. For example, in Figure 15.6, the line with slope -8 is steeper than the line with slope -1. A line with a negative slope near zero, such as the line in Figure 15.6 having slope $-\frac{1}{8}$, declines gradually from left to right.

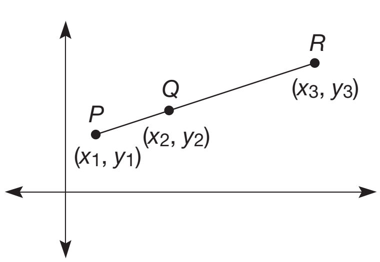


Figure 15.7

Slope and Collinearity Using slopes, we can determine whether several points are collinear. For example, if points P, Q, and R are collinear, then the slope of segment $\overline{PQ}$ is equal to the slope of segment $\overline{PR}$ (Figure 15.7). That is, if P, Q, and R are collinear, then $\dfrac{y_2-y_1}{x_2-x_1} = \dfrac{y_3-y_1}{x_3-x_1}$ provided that the slopes exist. If P, Q, and R are collinear and lie on a vertical line, then segments $\overline{PQ}$ and $\overline{PR}$ have no slope.

On the other hand, suppose that P, Q, and R are such that the slope of segment $\overline{PQ}$ is equal to the slope of segment $\overline{QR}$. It can be shown that the points P, Q, and R are collinear.

THEOREM

Slope and Collinearity

Points P, Q, and R are collinear if and only if

1. the slopes of $\overline{PQ}$ and $\overline{QR}$ are equal, or

2. the slopes of $\overline{PQ}$ and $\overline{QR}$ are undefined (i.e., P, Q, and R are on the same vertical line).

NCTM Standard
All students should use coordinate geometry to represent and examine the properties of geometric shapes.

Slopes of Parallel Lines In Figure 15.5 it appears that lines l and m, which have the same slope, are parallel. We can determine whether two lines are parallel by computing their slopes.

THEOREM

Slopes of Parallel Lines

Two lines in the coordinate plane are parallel if and only if

1. their slopes are equal, or

2. their slopes are undefined.

In Example 15.3, lines l and m each have slope $\frac{11}{18}$ and thus are parallel by the slopes of parallel lines theorem.

Slopes of Perpendicular Lines Just as we are able to determine whether lines are parallel using their slopes, we can also identify perpendicular lines by means of their slopes. For example, consider line l in Figure 15.8(a). Since l contains (0, 0) and (x_1, y_1), the slope of l is $\frac{y_1 - 0}{x_1 - 0} = \frac{y_1}{x_1}$. If line l is rotated 90° clockwise around (0, 0) to l', then l is perpendicular to l' [Figure 15.8(b)].

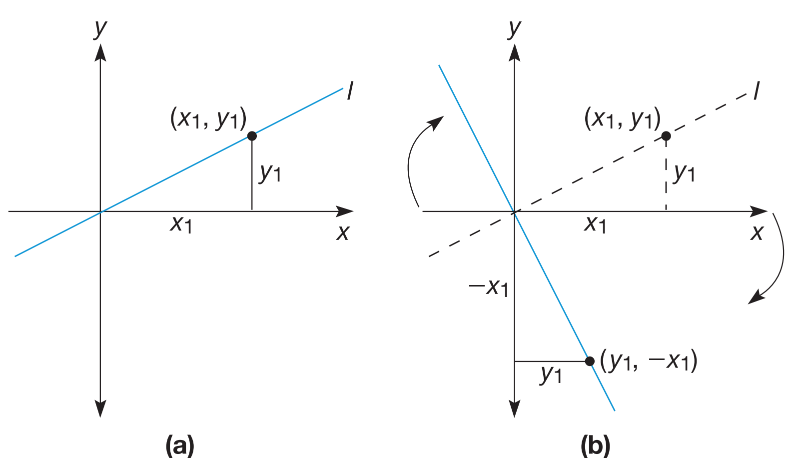


Figure 15.8

Also, point $(y_1, -x_1)$ is on l'. So the slope of l' is $\frac{-x_1 - 0}{y_1 - 0} = \frac{-x_1}{y_1}$. Therefore, the product of the slopes of the perpendicular lines l and l' is $\frac{y_1}{x_1} \times \frac{-x_1}{y_1} = -1$.

STUDENT PAGE SNAPSHOT

LESSON 2

Algebra: Graph Integers on the Coordinate Plane

Quick Review

Write the integer that is 1 less than the given integer.

1. $^{-}5$
2. $^{-}1$
3. $^{+}3$
4. 0
5. $^{-}6$

VOCABULARY

coordinate plane
origin
coordinates

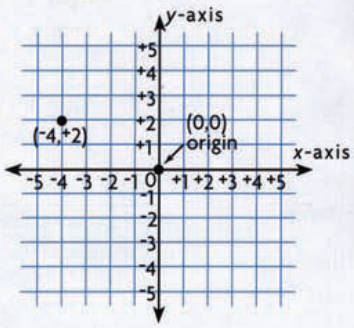
Learn

MAPPING HISTORY Archaeologists record the locations of artifacts and features that they find in a dig site by graphing them on a coordinate plane.

A **coordinate plane** is formed by two intersecting and perpendicular number lines called axes. The point where the two lines intersect is called the **origin**, or (0,0).

The numbers to the left of the origin on the *x*-axis and below the origin on the *y*-axis are negative.

Start at the origin. Move 4 units to the left on the *x*-axis and 2 units up on the *y*-axis. The **coordinates**, or numbers in the ordered pair, are $(^{-}4, ^{+}2)$.



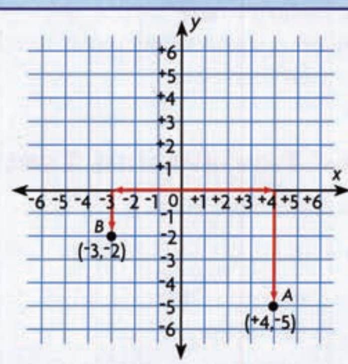
▲ Christopher Wolfe, at the age of 7, co-discovered the oldest horned ceratopsian dinosaur.

HANDS ON

Activity

MATERIALS: coordinate plane

- Graph the ordered pair $(^{+}4, ^{-}5)$. Start at the origin. Move **right** 4 units and then **down** 5 units. Plot and label the point, *A*.
- Graph the ordered pair $(^{-}3, ^{-}2)$. Start at the origin. Move **left** 3 units and then **down** 2 units. Plot and label the point, *B*.
- In which direction and how far would you move to graph $(^{-}4, ^{+}5)$?



MATH IDEA The coordinates of a point tell you how far and in which direction to move first horizontally and then vertically on the coordinate plane.

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Conversely, to show that *if* the product of the slopes of two lines is -1, *then* the lines are perpendicular, we can use a similar argument. This is left for Part B Problem 22 in the Problem Set. In summary, we have the following result.

NCTM Standard
All students should use coordinate geometry to examine special geometric shapes such as regular polygons or those with pairs of parallel or perpendicular sides.

THEOREM

Slopes of Perpendicular Lines

Two lines in the coordinate plane are perpendicular if and only if

1. the product of their slopes is -1, or

2. one line is horizontal and the other is vertical.

Example 15.4 shows how slopes can be used to analyze the diagonals of a rhombus.

Example 15.4 Show that the diagonals of *PQRS* in Figure 15.9 form right angles at their intersection.

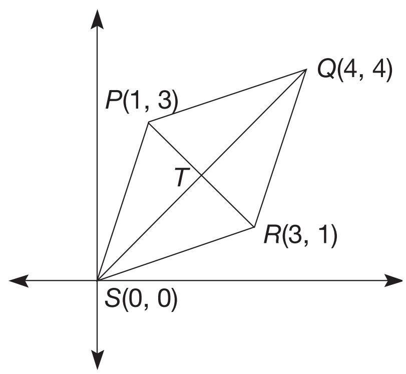


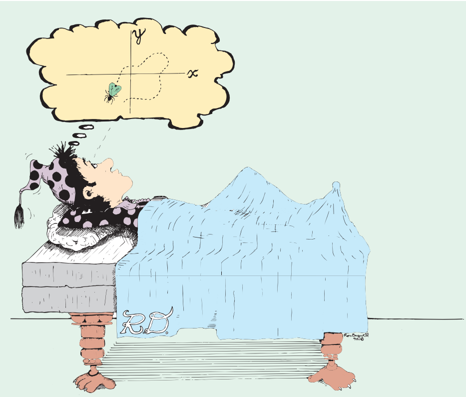
Figure 15.9

SOLUTION The slope of diagonal $\overline{PR}$ is $\frac{2}{-2} = -1$, and the slope of diagonal $\overline{QS}$ is $\frac{4}{4} = 1$.

Since the product of their slopes is -1, the diagonals form right angles at T by the slopes of perpendicular lines theorem. Each side of *PQRS* has length $\sqrt{10}$. Thus it is a rhombus. In fact, the result in Example 15.4 holds for *any* rhombus. This result is presented as a problem in Section 15.3. ■

MATHEMATICAL MORSEL

Descartes was creative in many fields: philosophy, physics, cosmology, chemistry, physiology, and psychology. But he is best known for his contributions to mathematics. He was a frail child of a noble family. As one story goes, due to his frailty, Descartes had a habit of lying in bed, thinking for extended periods. One day, while watching a fly crawling on the ceiling, he set about trying to describe the path of the fly in mathematical language. Thus was born analytic geometry—the study of mathematical attributes of lines and curves.



Section 15.1 EXERCISE / PROBLEM SET A

EXERCISES

1. Find the distance between the following pairs of points.
a. (0, 0), (3, −2) **b.** (−4, 2), (−2, 3)

2. Use the distance formula to determine whether points *P*, *Q*, and *R* are collinear.
a. *P*(−1, 4), *Q*(−2, 3), and *R*(−4, 1)
b. *P*(−2, 1), *Q*(3, 4), and *R*(12, 10)

3. The endpoints of a segment are given. Find the coordinates of the midpoint of the segment.
a. (0, 2) and (−3, 2)
b. (−5, −1) and (3, 5)
c. (−2, 3) and (−3, 6)
d. (3, −5) and (3, −7)

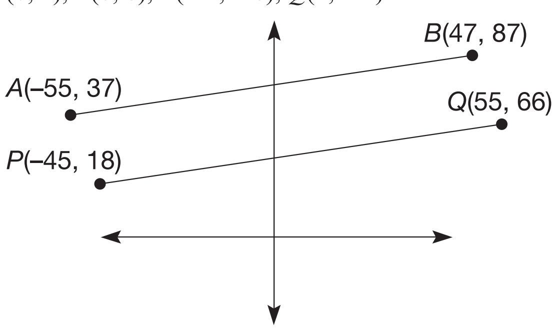
4. Find the slopes of the lines containing the following pairs of points.
a. (3, 2) and (5, 3)
b. (−2, 1) and (−5, −3)

5. Use the ratio $\dfrac{y_2 - y_1}{x_2 - x_1}$ and the ratio $\dfrac{y_1 - y_2}{x_1 - x_2}$ to compute the slopes of the lines containing the following points. Do both ratios give the same result?
a. (1, 4) and (5, 2)
b. (−2, −3) and (3, 2)

6. To gain a sense of how the numerical values of the slope are related to the visual appearance, use the Chapter 15 Geometer's Sketchpad® activity *Slope* on our Web site. Move the line to see the slope change dynamically. Describe what lines with the following slopes look like.
a. 5 **b.** $-\dfrac{1}{3}$ **c.** −7 **d.** $\dfrac{8}{9}$

7. Use slopes to determine whether the points *A*(3, −2), *B*(1, 2), and *C*(−3, 10) are collinear.

8. Use slopes to determine whether $\overline{AB} \parallel \overline{PQ}$.
a. *A*(−1, 0), *B*(4, 5), *P*(3, 9), *Q*(−2, 4)
b. *A*(0, 4), *B*(6, 8), *P*(−4, −6), *Q*(2, −2)
c.



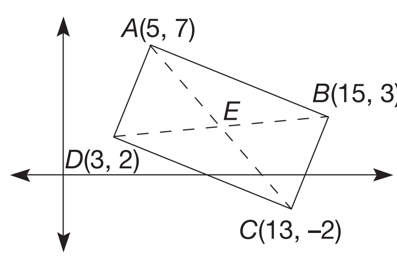
9. Determine whether the quadrilaterals with the given vertices are parallelograms.
a. (1, 4), (4, 4), (5, 1), and (2, 1)
b. (1, −1), (6, −1), (6, −4), and (1, −4)

10. Give the slope of a line perpendicular to $\overleftrightarrow{AB}$.
a. *A*(1, 6), *B*(2, 5)
b. *A*(0, 4), *B*(−6, −5)
c. *A*(4, 2), *B*(−5, 2)
d. *A*(−1, 5), *B*(−1, 3)

11. In each part, use the Chapter 15 eManipulative activity *Coordinate Geoboard* on our Web site to determine whether any of the line segments joining the given points are perpendicular.
a. (2, 2), (2, 4), and (5, 4)
b. (−3, 4), (0, −1), and (5, 2)
c. (−5, −4), (3, 2), and (5, 2)
d. (−2, −2), (1, 4), and (5, −4)

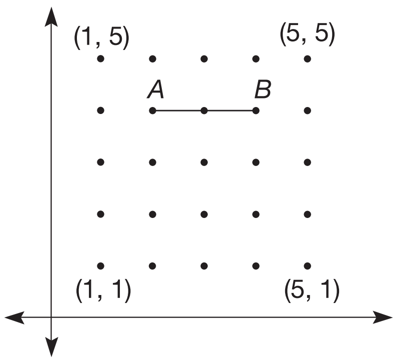
12. Determine which of the quadrilaterals, with the given vertices, is a rectangle.
a. (−10, −5), (−6, 15), (14, 11), and (10, −9)
b. (−2, 1), (2, 9), (6, 7), (2, −2)

13. Consider the following quadrilateral.



a. Verify that *ABCD* is a rectangle.
b. What do you observe about the lengths *BD* and *AC*?
c. What do you observe about $\overline{AE}$ and $\overline{CE}$ about $\overline{BE}$ and $\overline{DE}$?
d. Are $\overline{AC}$ and $\overline{BD}$ perpendicular? Explain.
e. Summarize the properties you have observed about rectangle *ABCD*.

14. Using *only* the array of points and points $A(2, 4)$ and $B(4, 4)$ as the endpoints of one side of a quadrilateral, answer the following questions. The Chapter 15 eManipulative *Coordinate Geoboard* on our Web site will help in solving this problem.



a. Use coordinates to list all possible parallelograms that can be constructed on the above array.
b. Use coordinates to list all possible rectangles.
c. Use coordinates to list all possible rhombi.
d. Use coordinates to list all possible squares.

15. On the Chapter 15 eManipulative activity *Coordinate Geoboard* on our Web site construct triangles that have vertices with the given coordinates. Describe the triangles as scalene, isosceles, equilateral, acute, right, or obtuse. Explain.
a. $(0, 0), (0, -4), (4, -4)$
b. $(-3, -1), (1, 2), (5, -1)$

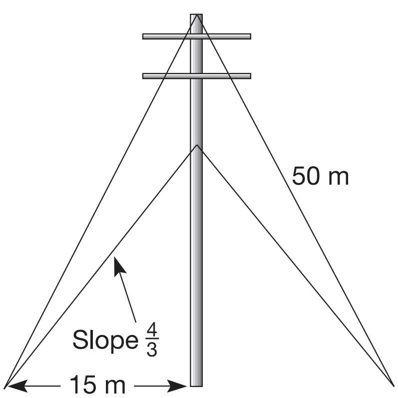
16. Given are the coordinates of the vertices of a triangle. Use the coordinate distance formula to determine whether the triangle is a right triangle.
a. $A(-2, 5), B(0, -1), C(12, 3)$
b. $D(2, 3), E(-2, -3), F(-6, 1)$

17. Draw the quadrilateral $ABCD$ whose vertices are $A(3, 0)$, $B(6, 6)$, $C(6, 9)$, and $D(0, 6)$. Divide each of the coordinates by 3 and graph the new quadrilateral $A'\ B'\ C'\ D'$. For example, A' has coordinates $(1, 0)$. How do the lengths of corresponding sides compare?

PROBLEMS

18. Generating many examples of perpendicular lines where the product of their slopes is -1 is easily done on the Chapter 15 Geometer's Sketchpad® activity *Perpendicular Lines* on our Web site. Use this activity to find the slopes of lines that are perpendicular to the lines with the given slopes below.
a. 0.941 **b.** -0.355 **c.** 9.625

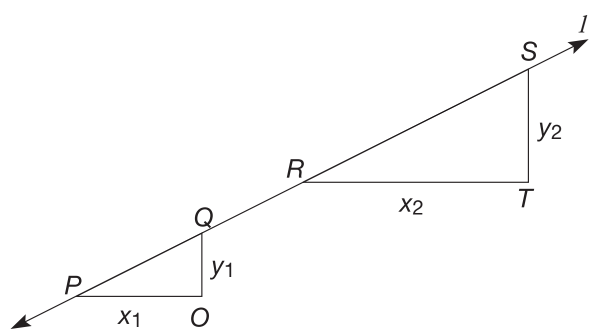
19. A pole is supported by two sets of guy wires fastened to the ground 15 meters from the pole. The shorter set of wires has slope $\pm\frac{4}{3}$. The wires in the longer set are each 50 meters long.



a. How high above the ground is the shorter set of wires attached?
b. What is the length of the shorter set of wires?
c. How tall is the pole to the nearest centimeter?
d. What is the slope of the longer set of wires to two decimal places?

20. A freeway ramp connects a highway to an overpass 10 meters above the ground. The ramp, which starts 150 meters from the overpass, is 150.33 meters long and 9 meters wide. What is the **percent grade** (rise/run expressed as a percent) of the ramp to three decimal places?

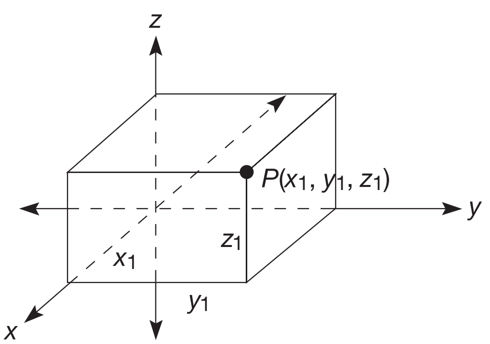
21. Let P, Q, R, and S be any points on line l as shown. By drawing horizontal and vertical segments, draw right triangles $\triangle PQO$ and $\triangle RST$. Follow the given steps to verify that the slope of line l is independent of the pairs of points selected.



a. Show that $\triangle PQO \sim \triangle RST$.
b. Show that $\frac{y_1}{x_1} = \frac{y_2}{x_2}$.
c. Is the slope of $\overline{PQ}$ equal to the slope of $\overline{RS}$? Explain.

22. In the development of the midpoint formula, points P and Q have coordinates (x_1, y_1) and (x_2, y_2), respectively, and M is the point $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$. Use the coordinate distance formula to verify that P, M, and Q are collinear and that $PM = MQ$.

23. Cartesian coordinates may be generalized to three-dimensional space. A point in space is determined by giving its location relative to three axes as shown. Point P with coordinates (x_1, y_1, z_1) is plotted by going x_1 along the x-axis, y_1 along the y-direction, and z_1 in the z-direction. Plot the following points in three-dimensional space.
a. (2, 1, 3) **b.** (−2, 1, 0) **c.** (3, −1, −2)



24. Janiece was trying to find the distance between (3, 7) and (9, 6) in the coordinate plane. She knew the formula was $D = \sqrt{(9-3)^2 + (6-7)^2}$. So she took the square roots and got $(9 - 3) + (6 - 7) = 5$. Did Janiece get the right answer? Will her method always work? Discuss.

25. Edmund was working on the same problem, but the numbers under the radical looked different: $D = \sqrt{(9-3)^2 + (7-6)^2}$. Shirdena told him he couldn't subtract the numbers that way; he had to subtract the coordinates in the same order. So he had to put 6 − 7, not 7 − 6. Who is correct? What explanation would you give to them?

Section 15.1 EXERCISE / PROBLEM SET B

EXERCISES

1. Find the distance between the following pairs of points.
a. (2, 3), (−1, −5)
b. (−3, 5), (−3, −2)

2. Use the distance formula to determine whether points P, Q, and R are collinear.
a. $P(-2, -3)$, $Q(2, -1)$, and $R(10, 3)$
b. $P(2, 7)$, $Q(-2, -7)$, and $R(3, 10.5)$

3. The point M is the midpoint of $\overline{AB}$. Given the coordinates of the following points, find the coordinates of the third point.
a. $A(-3, -1)$, $M(-1, 3)$
b. $B(-5, 3)$, $M(-7, 3)$
c. $A(1, -3)$, $M(4, 1)$
d. $M(0, 0)$, $B(-2, 5)$

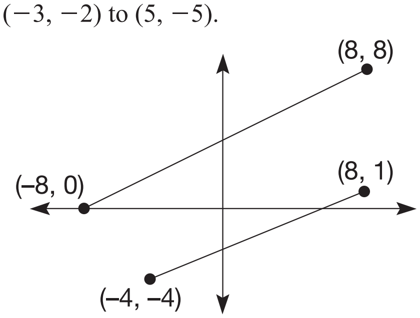
4. Find the slope of each line containing the following pairs of points.
a. (−1, 2) and (6, −3)
b. (6, 5) and (6, −2)

5. Use the ratio $\frac{y_2 - y_1}{x_2 - x_1}$ and the ratio $\frac{y_1 - y_2}{x_1 - x_2}$ to compute the slopes of the lines containing the following points. Do both ratios give the same result?
a. (−4, 5) and (6, −3) **b.** (1, −3) and (−2, −5)

6. Given are the slopes of several lines. Indicate whether each line is horizontal, vertical, rises to the right, or rises to the left.
a. $\frac{3}{4}$ **b.** no slope **c.** 0 **d.** $-\frac{5}{6}$

7. Use slopes to determine if the points $A(0, 7)$, $B(2, 11)$, and $C(-2, 1)$ are collinear.

8. Determine which pairs of segments are parallel.
a. The segment from (3, 5) to (8, 3) and the segment from (0, 8) to (8, 5).
b. The segment from (−4, 5) to (4, 2) and the segment from (−3, −2) to (5, −5).
c.



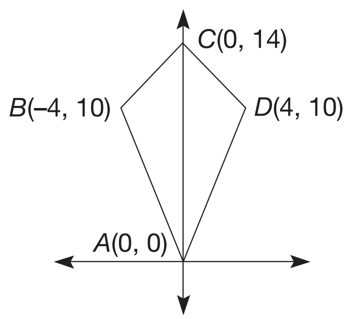
9. Determine whether the quadrilaterals with the given vertices are parallelograms.
a. (1, −2), (4, 2), (6, 2), and (3, −2)
b. (−10, 5), (−5, 10), (10, −5), and (5, −10)

10. Use slopes to show that $\overline{AB} \perp \overline{PQ}$.
  a. $A(0, 4)$, $B(-6, 3)$, $P(-2, -2)$, $Q(-3, 4)$
  b. $A(-2, -1)$, $B(2, 3)$, $P(4, 1)$, $Q(0, 5)$

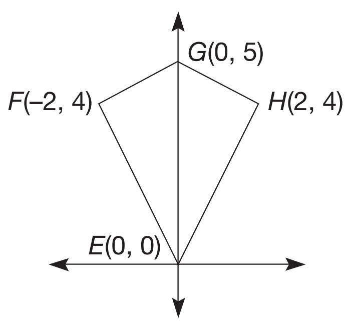
11. In each part, use the Chapter 15 eManipulative activity *Coordinate Geoboard* on our Web site to determine which, if any, of the triangles with the given vertices is a right triangle.
  a. $(-5, -4)$, $(-1, 4)$, and $(1, 2)$
  b. $(4, -4)$, $(0, 2)$, and $(-3, 0)$
  c. $(2, -3)$, $(4, 4)$, and $(-1, -2)$

12. Determine which of the quadrilaterals, with the given vertices, is a rectangle.
  a. $(0, 0)$, $(12, 12)$, $(16, 8)$, and $(4, -4)$
  b. $(-3, 8)$, $(0, 12)$, $(12, 3)$, and $(9, -1)$

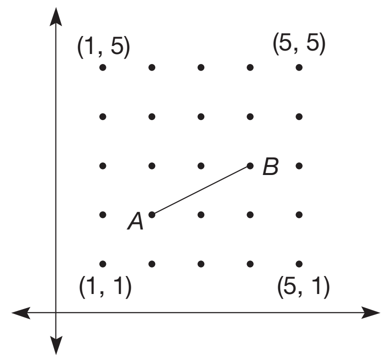
13. Which of the following properties are true of the given kites? Explain.
  a. The diagonals are congruent.
  b. The diagonals are perpendicular to each other.
  c. The diagonals bisect each other.
  d. The kite has two right angles.
  i.



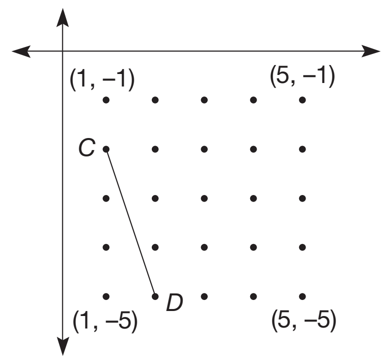
  ii.



14. Using only the 5 × 5 array of points, how many segments can be drawn making a right angle at an endpoint of the given segment? The Chapter 15 eManipulative *Coordinate Geoboard* on our Web site will help in solving this problem.
  a.



  b.



15. On the Chapter 15 eManipulative activity *Coordinate Geoboard* on our Web site construct triangles that have vertices with the given coordinates. Describe the triangles as scalene, isosceles, equilateral, acute, right, or obtuse. Explain.
  a. $(-3, 1)$, $(1, 3)$, $(5, -4)$ b. $(-4, -2)$, $(-1, 3)$, $(4, -2)$

16. Given are the coordinates of the vertices of a triangle. Use the coordinate distance formula to determine whether the triangle is a right triangle.
  a. $G(-3, -2)$, $H(5, -2)$, $I(1, 2)$ b. $L(1, 4)$, $M(5, 2)$, $N(4, 0)$

17. Draw a triangle ABC whose vertices are $A(8, 4)$, $B(4, 4)$, $C(8, 1)$. Multiply each of the coordinates by 2 and graph the new triangle $A'B'C'$. For example A' has coordinates $(16, 8)$.
  a. How do the lengths of the corresponding sides compare?
  b. How do the slopes of the corresponding sides compare?

PROBLEMS

18. What general type of quadrilateral is $ABCD$, where $A = (0, 0)$, $B = (-4, 3)$, $C = (-1, 7)$, and $D = (6, 8)$? Describe it as completely as you can.

19. Three of the vertices of a parallelogram have coordinates $P(2, 3)$, $Q(5, 7)$, and $R(10, -5)$.
  a. Find the coordinates of the fourth vertex of the parallelogram. (*Hint:* There is more than one answer.)
  b. Determine the area of each parallelogram.

20. a. Draw quadrilateral $ABCD$ where $A(4, -2)$, $B(4, 2)$, $C(-2, 2)$, and $D(-2, -2)$.
  b. Multiply each coordinate by 3 and graph quadrilateral $A'B'C'D'$. For example, A' has coordinates $(12, -6)$.
  c. How do the perimeters of $ABCD$ and $A'B'C'D'$ compare?
  d. How do the areas of $ABCD$ and $A'B'C'D'$ compare?
  e. Repeat parts (b) to (d), but divide each coordinate by 2.

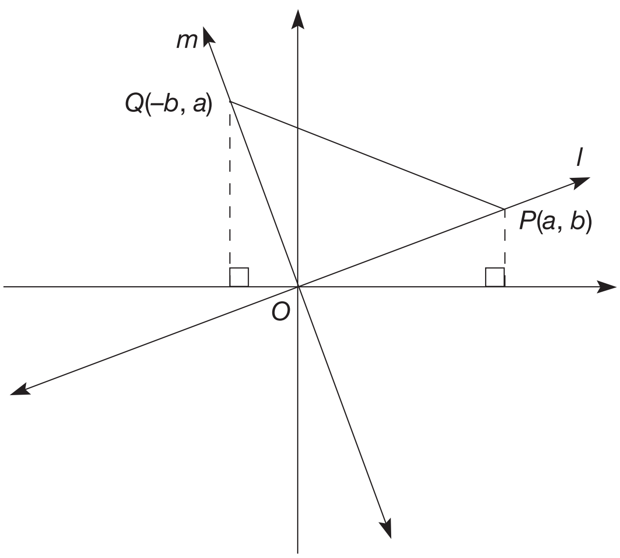
21. The percent grade of a highway is the amount that the highway rises (or falls) in a given horizontal distance. For example, a highway with a 4% grade rises 0.04 mile for every 1 mile of horizontal distance.

a. How many feet does a highway with a 6% grade rise in 2.5 miles?

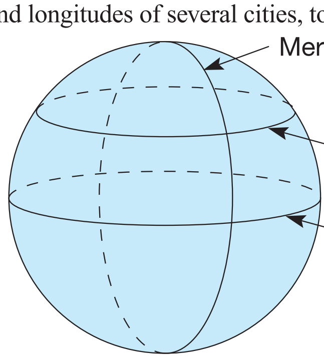
b. How many feet would a highway with a 6% grade rise in 90 miles if the grade remained constant?

c. How is percent grade related to slope?

22. Prove the following statement: If the product of the slopes of l and m is -1, lines l and m are perpendicular. (*Hint:* Show that $\triangle OPQ$ is a right triangle.)



23. Locations on the Earth's surface are measured by two sets of circles. **Parallels of latitude** are circles at constant distances from the equator used to identify locations north or south, called **latitude**. The latitude of the North Pole, for example, is 90°N, that of the equator is 0°, and that of the South Pole is 90°S. Great circles through the North and South poles, called **meridians**, are used to identify locations east or west, called **longitude**. The half of the great circle through the poles and Greenwich, England, is called the **prime meridian**. Points on the prime meridian have longitude 0°E. Points east of Greenwich and less than halfway around the world from it have longitudes between 0 and 180°E. Points west of Greenwich and less than halfway around the world from it have longitudes between 0 and 180°W. Here are the latitudes and longitudes of several cities, to the nearest degree.



CITY	LATITUDE	LONGITUDE
London	52°N	0°E
New York	41°N	74°W
Moscow	56°N	38°E
Nome, Alaska	64°N	165°W
Beijing	40°N	116°E
Rio de Janeiro	23°S	43°W
Sydney	34°S	151°E

a. Which cities are above the equator? Which are below?

b. Which city is farthest from the equator?

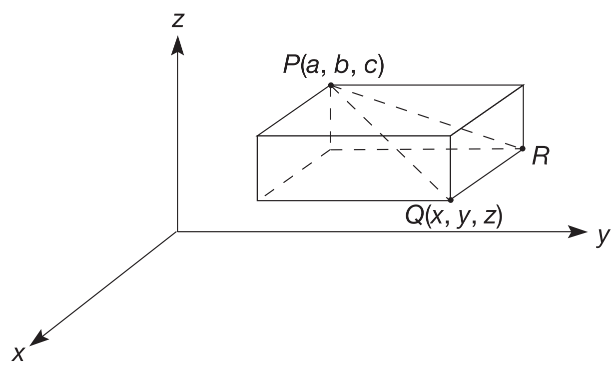
c. Are there points in the United States, excluding Alaska and Hawaii, with east longitudes? Explain.

d. What is the latitude of all points south of the equator and the same distance from the equator as London?

e. What is the longitude of a point exactly halfway around the world (north of the equator) from New York?

f. What are the latitude and longitude of a point diametrically opposite Sydney? That is, the point in question and Sydney are the endpoints of a diameter of the Earth.

24. The coordinate distance formula can also be generalized to three-dimensional space. Let P have coordinates (a, b, c) and Q have coordinates (x, y, z). The faces of the prism are parallel to the coordinate planes.



a. What are the coordinates of point R?

b. What is the distance PR? (*Hint:* Pythagorean theorem.)

c. What is the distance QR?

d. What is the distance PQ? (*Hint:* Pythagorean theorem.)

25. Shirdena was trying to find the slope of the segment whose endpoints were $(-2, 7)$ and $(-5, -8)$. She knew she had to subtract both numbers the same way, so she wrote $\frac{7 - (-8)}{-2 - (-5)}$. She got $\frac{15}{3}$, which she said equals 5. Edmund put $\frac{-8 - 7}{-5 - (-2)}$. He got $\frac{-15}{-3}$, which he said equals -5. Is it acceptable for them to get two different answers for this problem or is there something wrong here? Discuss.