Logic

In math, our reasoning is more strict than in everyday language. When a mother tells her child, "If you don't behave, you won't get any ice cream!" the child feels it is fair to assume that if he does behave, he will get ice cream. However, strictly speaking, the mother didn't promise him any ice cream for good behavior, she only promised no ice cream for bad behavior. A different example may illustrate this more clearly.

Let's assume the following statement is true: **"If it is nighttime, then you can't see the sun."** In our part of the world this is true. Now, the following are all variations of the statement. Which are equivalent to the original statement?

Converse: If you can't see the sun, then it is nighttime.

Inverse: If it is not nighttime, then you can see the sun.

Contrapositive: If you can see the sun, then it is not nighttime.

Some days it's cloudy – on these days you can't see the sun, but it's not nighttime. This contradicts two of the variations, so the converse and inverse above are not equivalent to the original statement. The contrapositive, however, is equivalent to the statement, and is true.

Caution: Even if a statement and its converse are both true, they are not equivalent. They are making different assertions. For example, "If I am hungry, it is time to eat" and "If it is time to eat then I am hungry" are both true statements for me :), but they are not equivalent. Suppose my husband announces "Supper is ready; you must be hungry!" Which of the two statements above justifies his claim? Suppose I complain, "I'm starving, it must be lunchtime!" Which of the two statements above justifies my complaint?

Sometimes these variations on a statement are summarized using the shorthand below:

Statement: If P, then Q
Converse: If Q, then P
Inverse: If not P, then not Q
Contrapositive: If not Q, then not P

Match up P and Q with the statements about the sun and nighttime above. Do you see how they match these phrases?

Practice: Write each of these variations for the statements below. In each case, note which variations of the original statement are true and which are not necessarily true. (In math, an "if—then—" statement is considered a **guarantee**. If the "if" portion is fulfilled, then the "then" part must follow. If a statement is "not necessarily true" or only "sometimes true," it is considered "false.")

Statement #1: If $x = 2$ then $x^2 = 4$.

Statement #2: If $2x = 6$ then $x = 3$.

For which of these statements did all of the variations turn out to be true?

When both a statement and its converse are true we have a very strong fact indeed. Special wording is often used to identify such a fact: the phrase "if and only if" (abbreviated "iff"). We can use this wording to summarize Statement #2 and its converse, as follows:

$2x = 6$ **if and only if** $x = 3$.

All definitions are considered to be "if and only if." Thus, saying "The definition of an isosceles triangle is a triangle with two or more congruent sides" means that both of the following statements are true:

"If a triangle has two or more congruent sides, then it is an isosceles triangle."
"If a figure is an isosceles triangle, then it is a triangle with two or more congruent sides."

Sometimes the most challenging step is rewriting a statement as an equivalent "If—then—" statement. Here are some examples of equivalent rewritten statements:

Example 1: Every rectangle is a quadrilateral.
Equivalent statement: If a figure is a rectangle then it is a quadrilateral.
NOT equivalent: If a figure is a quadrilateral then it is a rectangle.

Example 2: A figure can be a rhombus only if it has perpendicular diagonals.
Equivalent statement: If a figure is a rhombus then it has perpendicular diagonals.
NOT equivalent: If a figure has perpendicular diagonals then it is a rhombus.

Here are some statements to try re-writing.

If the given statement implies both "If P then Q" *and* "If Q then P," write an equivalent "if and only if" statement.

1. Buying supper at Boston Market always costs me $10.

2. All squares are rectangles.

3. You need to have at least a 3.3 GPA to be accepted into our PhD program.

4. The opposite angles of a rectangle are congruent.

5. Only upperclassman can have a car registered on campus.

6. Equilateral triangles are exactly those triangles with three congruent sides.

7. All squares, and only squares, are both rectangles and rhombuses.

Another perspective

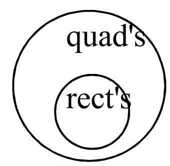
Make a Venn diagram showing the relationship between squares and rectangles. How is the Venn diagram related to the "If – then -- " statement? To the original statement in #2?

Make Venn diagrams for #3, 4, and 5. How is the Venn diagram related to the "If – then – " statement? To the original statement?

Necessary and Sufficient Conditions

Another way that mathematicians frequently describe relationships is to differentiate between "necessary" and "sufficient" conditions.

Change each statement below to an equivalent "if – then" statement.
Then draw a Venn diagram illustrating the relationship.



Example:

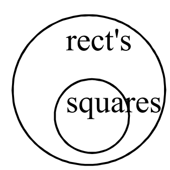
To be a rectangle, it is necessary that a figure be a quadrilateral.
If a figure is a rectangle, then it is a quadrilateral.

1. To be a rectangle, it is necessary that a figure have at least one right angle.

2. To get a driver's license it is necessary that you pass the written test.

3. To be a rectangle, it is necessary that a quadrilateral have four congruent angles.

Reflect: What patterns do you see?
For which does the Venn diagram seem inappropriate? Why?



Example:

To be a rectangle, it is sufficient that the figure be a square.
If a figure is a square, then it is a rectangle.

4. To be an isosceles triangle, it is sufficient that the figure be an equilateral triangle.

5. To have a place to live it is sufficient to buy a house.

6. To be a rectangle, it is sufficient to be a quadrilateral with four congruent angles.

Reflect: Which conditions above are both necessary and sufficient?
How does that relate to the "if-then" statements?
How does that relate to the Venn diagrams?