MATH 141, Tricks with Complex Numbers

Prof. Jonathan Rosenberg

Fall Semester, 2012

A very useful application of complex numbers is to simplify many of the integration formulas from Chapter 8 in Ellis and Gulick. Here are a few examples:

1. Suppose we want to compute $\int e^{ax}\cos(bx)\,dx$ or similar integrals. One method discussed in Chapter 8 is to use integration by parts twice, then solve for the unknown integral. This works, but it's complicated and time-consuming. The complex exponential function gives a much simpler approach. Recall that

$$e^{(a+ib)x} = e^{ax}\big(\cos(bx) + i\,\sin(bx)\big).$$

So if a, b, and x are real, $e^{ax}\cos(bx)$ is the real part of $e^{(a+ib)x}$. But integrating exponential functions is easy! So we get

$$\begin{aligned}
\int e^{ax}\cos(bx)\,dx &= \int \mathrm{Re}\Big(e^{(a+ib)x}\Big)\,dx \\
&= \mathrm{Re}\int e^{(a+ib)x}\,dx \\
&= \mathrm{Re}\Big(\frac{1}{a+ib}e^{(a+ib)x}\Big) + C \\
&= \mathrm{Re}\Big(\frac{a-ib}{a^2+b^2}e^{ax}\big(\cos(bx)+i\,\sin(bx)\big)\Big) + C \\
&= \frac{a}{a^2+b^2}e^{ax}\cos(bx) + \frac{b}{a^2+b^2}e^{ax}\sin(bx).
\end{aligned}$$

Done!

2. A rather mysterious result is the formula for the integral of the secant function: $\int \sec x\,dx = \ln(\sec x + \tan x) + C$ (for $|x| < \frac{\pi}{2}$). The standard derivation of this, found in Ellis and Gulick section 5.7, seems to come out of nowhere. A more natural approach uses the complex exponential function. Recall that in terms of complex exponentials, $\cos x = (e^{ix} + e^{-ix})/2$, and so $\sec x =$

$2/(e^{ix}+e^{-ix})$. So we obtain

$$\begin{aligned}
\int \sec x\,dx &= \int \frac{2}{e^{ix}+e^{-ix}}\,dx \\
&= \int \frac{2e^{ix}}{e^{ix}\big(e^{ix}+e^{-ix}\big)}\,dx \\
&= \int \frac{2e^{ix}}{1+e^{2ix}}\,dx \\
&\text{(substitute } u=e^{ix},\ du=ie^{ix}\,dx) \\
&= \int \frac{2\,du}{i(1+u^2)} \\
&= -2i\ \tan^{-1}(u)+C' = -2i\ \tan^{-1}\big(e^{ix}\big)+C'.
\end{aligned}$$

This looks rather strange, especially since if $x=0$, $e^{ix}=1$ and thus $-2i\tan^{-1}\big(e^{ix}\big)=-2i\tan^{-1}(1)=-i\,\pi/2$. So it's natural to relabel the constant and take $C'=\frac{i\pi}{2}+C$. This will make C real, and give

$$\int \sec x\,dx = -2i\ \tan^{-1}\big(e^{ix}\big)+\frac{i\pi}{2}+C.$$

This doesn't look at all like the usual formula, but it's possible to check that it's equivalent. For example, if $x=\pi/4$, $\ln(\sec x+\tan x)=\ln(1+\sqrt{2})$, while $e^{ix}=\cos x+i\ \sin x=\frac{1}{\sqrt{2}}+i\,\frac{1}{\sqrt{2}}=(1+i)/\sqrt{2}$. It turns out that

$$\tan^{-1}\left(\frac{1+i}{\sqrt{2}}\right)=\frac{\pi}{4}+i\,\frac{\ln(1+\sqrt{2})}{2},$$

so

$$-2i\ \tan^{-1}\left(\frac{1+i}{\sqrt{2}}\right)+\frac{i\pi}{2}=\frac{-i\pi}{2}+\frac{i\pi}{2}+2\frac{\ln(1+\sqrt{2})}{2}=\ln(1+\sqrt{2}),$$

the same as the usual formula.