Review Sheet on Convergence of Series
MATH 141H

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There are many tests for convergence of series, and frequently it can been confusing. How do you tell what test to use? Here's a quick run-down on the basics. We assume we have a given series

$$\sum_{n=k}^{\infty} a_n \qquad (*)$$

and want to know if it converges absolutely, conditionally, or not at all. Note that it doesn't matter that much what the initial value k of the index n is. In most cases it's 0 or 1, but it could be something else, and the question of convergence doesn't depend on k (though the *value* of the sum does depend on it).

n-th Term Test. This is the most basic test, but it usually doesn't help much. If a_n does not converge to 0 as $n \to \infty$, then the series $(*)$ does not converge. However, if $a_n \to 0$, this by itself doesn't tell you whether $(*)$ converges or not.

Comparison Test. If $\sum_{n=k}^{\infty} b_n$ converges, and if $b_n \geq 0$ for all n, and if $|a_n| \leq b_n$ for all sufficiently large n (it's OK if the inequality fails for *finitely many* values of n), then $(*)$ converges absolutely.

If $\sum_{n=k}^{\infty} b_n$ diverges, and if $b_n \geq 0$ for all n, and if $|a_n| \geq b_n$ for all sufficiently large n, then $(*)$ **does not** converge absolutely. (It may still converge conditionally, however; for that you frequently need to look at the Alternating Series Test.)

Limit Comparison Test. This is a slight modification of the above. If $\sum_{n=k}^{\infty} b_n$ converges, and if $b_n \geq 0$ for all n, and if $\lim_{n\to\infty}(|a_n|/b_n)$ exists and is finite, then $(*)$ converges absolutely. If $\sum_{n=k}^{\infty} b_n$ diverges, and if $b_n \geq 0$ for all n, and if $\lim_{n\to\infty}(|a_n|/b_n)$ exists and is bigger than 0, then $(*)$ **does not** converge absolutely.

Ratio Test. This is the test most often used to find the radius of convergence of a power series. If $\lim_{n\to\infty} |a_{n+1}/a_n| = r$ and $r < 1$, then $(*)$ converges absolutely. If $r > 1$ ($r = \infty$ is included), then $(*)$ diverges. If $r = 1$, then the test is inconclusive.

Root Test. This test is also sometimes used to find the radius of convergence of a power series. If $\lim_{n\to\infty} |a_n|^{1/n} = r$ and $r < 1$, then $(*)$ converges absolutely. If $r > 1$ ($r = \infty$ is included, then $(*)$ diverges. If $r = 1$, then the test is inconclusive.

Integral Test. Suppose $|a_n| = f(n)$, where f is a positive function that decreases to 0. If

$$\int_k^\infty f(x)\,dx < \infty,$$

then $(*)$ converges absolutely. If

$$\int_k^\infty f(x)\,dx = +\infty,$$

then $(*)$ does not converge absolutely, though it may still converge conditionally.

***p*-Series Test.** A special case of the integral test, worth remembering by itself, is sometimes called the p-Series Test. If $a_n = 1/n^p$ (with $p > 0$), then the series converges if $p > 1$ and diverges if $p \leq 1$. The reason is that

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{c\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^c = \begin{cases} \frac{1}{p-1}, & p > 1, \\ \infty, & 0 < p < 1 \end{cases},$$

unless $p = 1$, in which case

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{c\to\infty} [\ln x]_1^c = \infty.$$

So the improper integral converges for $p > 1$ and diverges for $p \leq 1$.

Alternating Series Test. This test is rather special, but is often useful in testing for conditional convergence. Suppose that, *after perhaps discarding finitely many terms at the beginning of the series*, $|a_n|$ decreases to 0 and the signs of the a_n alternate between $+$ and $-$. Then $(*)$ is convergent. Whether or not it is absolutely convergent needs to be checked with a different test.

Use of the Remainder Formula. Finally, there is one other technique that can sometimes be used. Suppose that $(*)$ is actually the Taylor series of a function,

$$\sum_{n=0}^\infty \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

for some smooth function f and some a and evaluated at x. Just as an example, the alternating harmonic series,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

is the Taylor series for $\ln(1+x)$ evaluated at $x = 1$. Then the series converges to $f(x)$ provided $R_n(x) \to 0$. For example, in this case, we have $f^{(n)}(x) = (-1)^{n+1}(n-1)!(x+1)^{-n}$ for $n \geq 1$, so

$$R_n(1) = \frac{1^{n+1}}{(n+1)!} f^{(n+1)}(t)$$

for some $0 \leq t \leq 1$, and since $(t+1)^{-n-1}$ is a decreasing function of t for $0 \leq t \leq 1$,

$$|R_n(1)| = \frac{1}{(n+1)!} n!(t+1)^{-n-1} \leq \frac{1}{(n+1)} \to 0,$$

so the series converges to $f(1) = \ln(1+1) = \ln 2$.

Another example. Show that the series

$$\sum_{n=1}^{\infty} \frac{n}{4^{n-1}} = \frac{1}{1} + \frac{2}{4} + \frac{3}{16} + \cdots$$

converges, and find the value of the sum.

Solution. You can check convergence with the ratio test, since

$$\frac{a_{n+1}}{a_n} = \left(\frac{n+1}{4^n}\right)\left(\frac{4^{n-1}}{n}\right) = \left(\frac{n+1}{n}\right)\left(\frac{4^{n-1}}{4^n}\right) = \frac{1}{4}\left(1 + \frac{1}{n}\right) \to \frac{1}{4} < 1.$$

But to find the value of the sum, we need to write the series as a Taylor series and use the remainder formula. If we replace $1/4$ by x, we are led to looking at the power series

$$f(x) = \sum_{n=1}^{\infty} n x^{n-1},$$

which looks vaguely familiar. Indeed, if we integrate term by term, that will replace x^{n-1} by x^n/n, and the $(1/n)$ factors will kill off the n's. So let's compute

$$\int_0^x f(x)\, dx = \sum_{n=1}^{\infty} n \int_0^x x^{n-1}\, dx = \sum_{n=1}^{\infty} n \frac{x^n}{n} = \sum_{n=1}^{\infty} x^n,$$

which we recognize as a geometric series. So $\int_0^x f(x)\,dx$ is the series for

$$\frac{1}{1-x}-1=\frac{1}{1-x}-\frac{1-x}{1-x}=\frac{x}{1-x},$$

since we are starting with $n=1$ and not with $n=0$. That tells us that $f(x)$ should be

$$\frac{d}{dx}\left(\frac{x}{1-x}\right)=\frac{(1-x)-(x)(-1)}{(1-x)^2}=\frac{1}{(1-x)^2}=(1-x)^{-2}.$$

Indeed, computing the Taylor series of this function (a binomial series) shows this is right. So we expect the series to converge to

$$f\left(\frac{1}{4}\right)=\frac{1}{\left(\frac{3}{4}\right)^2}=\frac{16}{9}=1.777777\cdots,$$

and indeed we see that the first few partial sums are

$$\begin{aligned}
s_1&=\frac{1}{1}=1,\\
s_2&=\frac{1}{1}+\frac{2}{4}=1.5,\\
s_3&=\frac{1}{1}+\frac{2}{4}+\frac{3}{16}=1.6875,\\
s_4&=\frac{1}{1}+\frac{2}{2}+\frac{3}{16}+\frac{4}{64}=1.75,\\
s_5&=\frac{1}{1}+\frac{2}{2}+\frac{3}{16}+\frac{4}{64}+\frac{5}{256}=1.76953125,
\end{aligned}$$

which seems to be converging to $1.777777\cdots$.

Now let's use the Remainder Formula to **prove** that the series converges to 16/9. Recall we have $f(x)=(1-x)^{-2}$ with Taylor series $\sum_{n=1}^{\infty} nx^{n-1}$, and we want to show the Taylor series converges to the value of the function when $x=1/2$. We have

$$\begin{aligned}
f(x)&=(1-x)^{-2},\\
f'(x)&=2(1-x)^{-3},\\
f''(x)&=6(1-x)^{-4},\\
f^{(n)}(x)&=(n+1)!(1-x)^{-n-2}.
\end{aligned}$$

So

$$R_n(x)=\frac{x^{n+1}}{(n+1)!}f^{(n+1)}(t)=\frac{x^{n+1}}{(n+1)!}(n+2)!(1-t)^{-n-3}=(n+2)\frac{x^{n+1}}{(1-t)^{n+3}}$$

for some t between 0 and x. We are interested in the case $x = 1/4$, which is bigger than 0. Since negative powers of $1-t$ *increase* as t increases, we can bound the expression above by what we get when we replace $1-t$ by $1-x = 3/4$. So

$$\begin{aligned}
R_n\left(\frac{1}{4}\right) &= (n+2)\frac{1}{4^{n+1}}(1-t)^{-n-3} \\
&\le (n+2)\frac{1}{4^{n+1}}\left(\frac{3}{4}\right)^{-n-3} \\
&= (n+2)\left(\frac{4^{n+3}}{4^{n+1}3^{n+3}}\right) = (n+2)\left(\frac{16}{3^{n+3}}\right) \\
&= 16\left(\frac{n+2}{3^{n+3}}\right) \to 0,
\end{aligned}$$

and the series converges to $f(1/4) = 16/9$, as desired.

Another Solution. Here's another solution, that doesn't use the remainder formula. As before, check for absolute convergence using the ratio test. Since the series converges absolutely, it is legitimate to rearrage it. (Remember: you **cannot** rearrange a conditionally convergent series and expect to get the same value for the sum.) So we can write:

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{n}{4^{n-1}} &= \frac{1}{1}+\frac{2}{4}+\frac{3}{16}+\cdots \\
&= \left(\frac{1}{1}+\frac{1}{4}+\frac{1}{16}+\cdots\right)+\left(\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\cdots\right)+\left(\frac{1}{16}+\frac{1}{64}+\cdots\right)+\cdots \\
&= \sum_{n=0}^{\infty}\frac{1}{4^n}+\sum_{n=1}^{\infty}\frac{1}{4^n}+\sum_{n=2}^{\infty}\frac{1}{4^n}+\cdots.
\end{aligned}$$

Now use the formula for the sum of a geometric series. We have $\sum_{n=0}^{\infty} x^n = (1-x)^{-1}$ for $|x| < 1$, and in particular, for $x = 1/4$. So for $k \ge 0$,

$$\sum_{n=k}^{\infty}\frac{1}{4^n} = \frac{1}{4^k}\sum_{n=0}^{\infty}\frac{1}{4^n} = \left(\frac{1}{4^k}\right)\left(1-\frac{1}{4}\right)^{-1} = \left(\frac{1}{4^k}\right)\left(\frac{4}{3}\right).$$

Thus

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{n}{4^{n-1}} &= \sum_{n=0}^{\infty} \frac{1}{4^n} + \sum_{n=1}^{\infty} \frac{1}{4^n} + \sum_{n=2}^{\infty} \frac{1}{4^n} + \cdots \\
&= \sum_{k=0}^{\infty} \left(\frac{4}{3}\right) \left(\frac{1}{4^k}\right) \\
&= \frac{4}{3} \sum_{k=0}^{\infty} \frac{1}{4^k} = \left(\frac{4}{3}\right) \left(\frac{4}{3}\right) = \frac{16}{9}.
\end{aligned}
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