Proof that $\sum n^{-2} = \pi^2/6$

Euler discovered the result in the title. The following gives a fairly elementary proof. We start with some preliminary results (Lemmas 1 and 2). The main steps in the proof are Lemmas 3, 4, and 5.

Let
$$g(x) = \frac{d}{dx}\left(\frac{x}{\sin x}\right) = \frac{d}{dx} x \csc x = \csc x - x \csc x \cot x.$$
Note that $g(0)$ is not defined. However,
$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \csc x - x \csc x \cot x = \lim_{x\to 0} \frac{\sin x - x\cos x}{\sin^2 x} = 0,$$
where the last limit is evaluated using l'Hôpital's rule. Therefore, we can define $g(0) = 0$ and $g(x)$ becomes a continuous function in the interval $0 \le x \le \pi/2$. Let M be the maximum value of $|g(x)|$ in this interval (it can be shown that $0 \le g(x) \le 1$, but we don't need this).

Lemma 1. *Let n be an integer. Then*
$$\left| \int_0^{\pi/2} g(x)\, \cos\left((2n+1)x\right)\, dx \right| \le \frac{M\pi}{2}.$$

Proof. For any function $F(x)$, we have $\left|\int_a^b F(x)\, dx\right| \le \int_a^b |F(x)|\, dx$. This happens because the positive and negative parts partially cancel each other in the integral of $F(x)$ but they do not cancel in the integral of $|F(x)|$. Applying this inequality to the integral in the statement of the lemma yields
$$\left| \int_0^{\pi/2} g(x)\, \cos\left((2n+1)x\right)\, dx \right| \le \int_0^{\pi/2} \left| g(x)\, \cos\left((2n+1)x\right)\right|\, dx \le \int_0^{\pi/2} M\, dx.$$
Since the last integral equals $M\pi/2$, we are done. □

Let
$$f_n(x) = 1 + 2\cos(2x) + 2\cos(4x) + 2\cos(6x) + \cdots + 2\cos(2nx).$$

Lemma 2. *If x/π is not an integer, then*
$$f_n(x) = \frac{\sin\left((2n+1)x\right)}{\sin x}.$$

Proof. This can be proved by trig identities, but it is easier to use $\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$. Therefore,
$$\begin{aligned} f_n(x) &= 1 + \left(e^{2ix} + e^{-2ix}\right) + \left(e^{4ix} + e^{-4ix}\right) + \cdots + \left(e^{2nix} + e^{-2nix}\right) \\ &= -1 + \left(1 + e^{2ix} + e^{4ix} + \cdots + e^{2nix}\right) + \left(1 + e^{-2ix} + e^{-4ix} + \cdots + e^{-2nix}\right). \end{aligned}$$

Recall that $1+r+r^2+\cdots+r^m = (r^{m+1}-1)/(r-1)$. This yields (use $r = e^{2ix}$ and $r = e^{-2ix}$)

$$\begin{aligned} f_n(x) &= -1 + \frac{e^{2ix(n+1)}-1}{e^{2ix}-1} + \frac{e^{-2ix(n+1)}-1}{e^{-2ix}-1} \\ &= -1 + \frac{e^{(2n+1)ix}-e^{-ix}}{e^{ix}-e^{-ix}} + \frac{e^{-(2n+1)ix}-e^{+ix}}{e^{-ix}-e^{+ix}}, \end{aligned}$$

where we multiplied the numerator and denominator of the first fraction by e^{-ix} and multiplied the numerator and denominator of the second fraction by e^{ix}. A little algebra combines the fractions to yield

$$f_n(x) = \frac{e^{(2n+1)ix}-e^{-(2n+1)ix}}{e^{ix}-e^{-ix}} = \frac{\sin((2n+1)x)}{\sin x},$$

where we have used the fact that $\sin x = \left(e^{ix}-e^{-ix}\right)/2i$. □

Let

$$E_n = \int_0^{\pi/2} x\, f_n(x)\, dx.$$

Lemma 3.

$$E_n = \frac{\pi^2}{8} - \sum_{\substack{1 \le j \le n \\ j \text{ odd}}} \frac{1}{j^2}.$$

Proof. When $j > 0$, integration by parts yields

$$\int_0^{\pi/2} x\, \cos(2jx)\, dx = \begin{cases} -1/(2j^2), & j \text{ odd}, \\ 0 & j \text{ even}. \end{cases}$$

Therefore, using the sum defining $f_n(x)$, we obtain

$$E_n = \int_0^{\pi/2} x\, dx \; + \sum_{j=1}^{n} \int_0^{\pi/2} 2x\, \cos(2jx)\, dx = \frac{\pi^2}{8} - \sum_{\substack{1 \le j \le n \\ j \text{ odd}}} \frac{1}{j^2}.$$

□

Lemma 4. $\lim_{n\to\infty} E_n = 0$.

Proof. Use the expression for $f_n(x)$ from Lemma 2 in the definition for E_n and integrate by parts to obtain

$$\begin{aligned} E_n &= \int_0^{\pi/2} x\, \frac{\sin((2n+1)x)}{\sin x}\, dx = \int_0^{\pi/2} \frac{x}{\sin x}\, \sin((2n+1)x)\, dx \\ &= -\frac{x}{\sin x}\, \frac{\cos((2n+1)x)}{2n+1}\Bigg|_0^{\pi/2} + \int_0^{\pi/2} g(x)\, \frac{\cos((2n+1)x)}{2n+1}\, dx \\ &= 0 + \frac{1}{2n+1} + \frac{1}{2n+1}\int_0^{\pi/2} g(x)\, \frac{\cos((2n+1)x)}{2n+1}\, dx. \end{aligned}$$

Note that we evaluated $x/\sin x$ at $x = 0$ by using $\lim_{x\to 0} x/\sin x = 1$.

Therefore

$$|E_n| \leq \frac{1}{2n+1} + \frac{1}{2n+1}\left|\int_0^{\pi/2} g(x)\, \cos\left((2n+1)x\right)\, dx\right| \leq \frac{1}{2n+1} + \frac{1}{2n+1}\frac{M\pi}{2},$$

by Lemma 1. This implies that $\lim_{n\to\infty} E_n = 0$. □

Lemma 5.

$$\sum_{\substack{1 \leq j < \infty \\ j \text{ odd}}} \frac{1}{j^2} = \frac{\pi^2}{8}.$$

Proof. By definition, the infinite sum is the limit of its partial sums. By Lemma 3, the difference between the nth partial sum and $\pi^2/8$ is E_n. By Lemma 4, $\lim E_n = 0$. Therefore, the limit of the partial sums is $\pi^2/8$. □

Theorem.

$$\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}.$$

Proof.

$$\frac{3}{4}\sum_{j=1}^{\infty}\frac{1}{j^2} \quad = \quad \left(1 - \frac{1}{4}\right)\sum_{j=1}^{\infty}\frac{1}{j^2} \quad = \quad \sum_{j=1}^{\infty}\frac{1}{j^2} \; - \; \sum_{j=1}^{\infty}\frac{1}{(2j)^2}.$$

This is the sum over all positive integers minus the sum over the even integers. What remains is the sum over the odd integers, so

$$\frac{3}{4}\sum_{j=1}^{\infty}\frac{1}{j^2} = \sum_{\substack{1 \leq j < \infty \\ j \text{ odd}}} \frac{1}{j^2} = \frac{\pi^2}{8}.$$

Multiplying both sides by 4/3 yields the result. □

Where does this proof come from? Let $F(x)$ be the function $2x/\pi$ for $0 \leq x \leq \pi/2$ and $2 - (2/\pi)x$ for $\pi/2 \leq x \leq \pi$, so the graph of $F(x)$ is a line from $(0,0)$ to $(\pi/2, 1)$, then is a line back down to $(\pi, 0)$. It is possible to write

$$F(x) = \frac{\pi^2}{8} - \sum_{\substack{1 \leq j < \infty \\ j \text{ odd}}} \frac{1}{j^2}\cos(2jx).$$

The sum on the right is an example of what is known as a Fourier series. Evaluating at $x = 0$ yields Lemma 5. The proof of Lemma 4 is a special case of the proof that $F(x)$ equals the Fourier series for all values of x.