The Integral Form of the Remainder in Taylor's Theorem MATH 141H

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Let f be a smooth function near $x = 0$. For x close to 0, we can write $f(x)$ in terms of $f(0)$ by using the Fundamental Theorem of Calculus:

$$f(x) = f(0) + \int_0^x f'(t)\,dt.$$

Now integrate by parts, setting $u = f'(t)$, $du = f''(t)\,dt$, $v = t - x$, $dv = dt$. (Remember, the variable of integration is t, and we're thinking of x as a constant.) We get

$$\begin{aligned} f(x) &= f(0) + \int_0^x f'(t)\,dt \\ &= f(0) + \big[(t-x)f'(t)\big]_{t=0}^{t=x} - \int_0^x (t-x)f''(t)\,dt \\ &= f(0) + xf'(0) - \int_0^x (t-x)f''(t)\,dt. \end{aligned}$$

Now repeat the process. Again, integrate by parts, this time with $u = f''(t)$, $du = f'''(t)\,dt$, $v = (t-x)^2/2$, $dv = (t-x)\,dt$. We get

$$\begin{aligned} f(x) &= f(0) + xf'(0) - \int_0^x (t-x)f''(t)\,dt \\ &= f(0) - \Big[\frac{(t-x)^2}{2} f''(t)\Big]_{t=0}^{t=x} + \int_0^x \frac{(t-x)^2}{2} f'''(t)\,dt \\ &= f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \int_0^x \frac{(t-x)^2}{2} f'''(t)\,dt. \end{aligned}$$

Continuing this process over and over, we see eventually that

$$f(x) = f(0) + xf'(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + R_n(x)$$

where the remainder $R_n(x)$ is given by the formula

$$R_n(x) = (-1)^n \int_0^x \frac{(t-x)^n}{n!} f^{(n+1)}(t)\, dt = \int_0^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt.$$

In principle this is an exact formula, but in practice it's usually impossible to compute. However, let's assume for simplicity that $x > 0$ (the case $x < 0$ is similar) and assume that

$$a \le f^{(n+1)}(t) \le b, \qquad 0 \le t \le x.$$

In other words, a is a lower bound for $f^{(n+1)}(t)$ on the interval $[0, x]$, and b is an upper bound for $f^{(n+1)}(t)$ on the same interval. Then we get

$$\int_0^x \frac{(x-t)^n}{n!} a\, dt \le R_n(x) = \int_0^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt \le \int_0^x \frac{(x-t)^n}{n!} b\, dt. \qquad (**)$$

But

$$\begin{aligned} \int_0^x \frac{(x-t)^n}{n!}\, dt &= (-1)^n \int_0^x \frac{(t-x)^n}{n!}\, dt = (-1)^n \Big[\frac{(t-x)^{n+1}}{(n+1)!}\Big]_{t=0}^{t=x} \\ &= (-1)^n \Big[\frac{0}{(n+1)!} - \frac{(-x)^{n+1}}{(n+1)!}\Big] = -(-1)^n(-1)^{n+1}\frac{x^{n+1}}{(n+1)!} \\ &= \frac{x^{n+1}}{(n+1)!}. \end{aligned}$$

Plugging this into $(**)$, we see that

$$a\frac{x^{n+1}}{(n+1)!} \le R_n(x) \le b\frac{x^{n+1}}{(n+1)!},$$

which is Lagrange's estimate for the remainder.