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R is a commutative ring. Suppose $p(x) \in R[x]$ is a zero divisor. Then there exists a nonzero constant $c \in R$ such that $cp(x) = 0$.

Proof. Let $q(x) \in R[x]$ be a nonzero polynomial of minimum degree such that $pq = 0$. Note that if the statement is correct then the degree of q is zero. We have $p(x) = a_n x^n + \cdots + a_0$ and $q(x) = b_m x^m + \cdots + b_0$. We assume that a_n and b_m are nonzero.

Suppose that $a_k b_l = 0$ for every $k \in \{0, ..., n\}$ and every $l \in \{0, ..., m\}$. Then $b_m p(x) = 0$ and we're finished.

Suppose that k' and l' are such that $a_{k'} b_{l'} \neq 0$. Let k be the least integer such that $a_{n-k} b_l \neq 0$ for some l. The coefficient of the x^{n+m-k} term of $p(x)q(x)$ is

$$a_n b_{m-k} + a_{n-1} b_{m-k+1} + \cdots + a_{n-k+1} b_{m-1} + a_{n-k} b_m = 0.$$

Since $a_{n-i} b_j = 0$ for all $i < k$ and for all $j \in \{0, ..., m\}$, we see that the first k terms of the above sum are each zero, and so therefore $a_{n-k} b_m = 0$. Note that $a_{n-k} q(x)$ is nonzero since $a_{n-k} b_l \neq 0$. Also, the degree of $a_{n-k} q(x)$ is less than the degree of $q(x)$, since $a_{n-k} b_m = 0$. In addition $p(x)(a_{n-k} q(x)) = 0$. This contradicts the minimality of the degree of q.