MATH 601: Abstract Algebra II
2nd Homework Solutions
Additional exercises (The functor Ext)

Jonathan Rosenberg

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If R is a ring, M is an R-module and Hom denotes Hom in the category of R-modules, then as pointed out in Hungerford and in class, the functor $\mathrm{Hom}(M, \_)$ is only left exact. In other words, if

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

is an exact sequence of R-modules, then

$$0 \to \mathrm{Hom}(M, A) \xrightarrow{\alpha_*} \mathrm{Hom}(M, B) \xrightarrow{\beta_*} \mathrm{Hom}(M, C)$$

is exact but the map $\mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C)$ is not necessarily onto, unless M is projective. But one can extend the sequence to the right as follows. Choose a free (or projective) R-module P_1 mapping onto M and another free (or projective) R-module P_2 mapping onto the kernel of the map $P_1 \to M$, etc. That gives an exact sequence

$$P_3 \xrightarrow{\varepsilon} P_2 \xrightarrow{\delta} P_1 \xrightarrow{\gamma} M \to 0.$$

Define $\mathrm{Ext}^1_R(M, A)$ to be the kernel of

$$\mathrm{Hom}(P_2, A) \xrightarrow{\varepsilon^*} \mathrm{Hom}(P_3, A)$$

divided by the image of

$$\mathrm{Hom}(P_1, A) \xrightarrow{\delta^*} \mathrm{Hom}(P_2, A).$$

1. Show that $\mathrm{Ext}^1_R(M, A) = 0$ if M is R-projective.
Solution. If M is R-projective, we can simply take $P_1 = M$, $\gamma = \mathrm{id}_M$, and $P_2 = P_3 = 0$. Then clearly the above definition (assuming it's well defined) gives $\mathrm{Ext}^1_R(M, A) = 0$. $\square$

2. Show that one gets an exact sequence

$$0 \to \mathrm{Hom}(M, A) \xrightarrow{\alpha_*} \mathrm{Hom}(M, B) \xrightarrow{\beta_*} \mathrm{Hom}(M, C) \to \mathrm{Ext}^1_R(M, A).$$

The proof is a long diagram chase.

Solution. We already know that α^* is injective and that $\ker \beta_* = \operatorname{im} \alpha^*$. So the problem is to define the map $\mathrm{Hom}(M, C) \to \mathrm{Ext}^1_R(M, A)$ and to prove exactness at $\mathrm{Hom}(M, C)$. Given $f\colon M \to C$, compose with γ to get $f \circ \gamma\colon P_1 \to C$. By projectivity of P_1, this lifts to a map $g\colon P_1 \to B$, with $\beta \circ g = f \circ \gamma$. In other words, we have a commuting diagram with exact rows:

$$\begin{tikzcd}
P_3 \arrow[r, "\varepsilon"] & P_2 \arrow[r, "\delta"] \arrow[d, dotted, "h"] & P_1 \arrow[r, "\gamma"] \arrow[d, "g"] & M \arrow[r] \arrow[d, "f"] & 0 \arrow[d, equal] \\
0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 0
\end{tikzcd}$$

and try to fill in with a map $h\colon P_2 \to A$ as shown. Indeed, since $\gamma \circ \delta = 0$, $0 = f \circ \gamma \circ \delta = \beta \circ g \circ \delta$, so $g \circ \delta\colon P_2 \to B$ lands in $\ker \beta = \operatorname{im} \alpha$, and since α is injective, we can fill in with h. Furthermore, $h \circ \varepsilon = 0$, since $\alpha \circ h \circ \varepsilon = g \circ \delta \circ \varepsilon = 0$, so $h \in \ker\bigl(\mathrm{Hom}(P_2, A) \xrightarrow{\varepsilon^*} \mathrm{Hom}(P_3, A)\bigr)$ and defines a class in $\mathrm{Ext}^1_R(M, A)$. We can also see that this class is well defined, because any two choices for g differ by a map $P_1 \to \ker \beta = \operatorname{im} \alpha$, and result in h changing by something in the image of $\mathrm{Hom}(P_1, A) \xrightarrow{\delta^*} \mathrm{Hom}(P_2, A)$.

Now we only need to check exactness. Suppose h is trivial in $\mathrm{Ext}^1_R(M, A)$, i.e., lies in the image of δ^*. That means we have $k\colon P_1 \to A$ with $h = k \circ \delta$. Then since $\alpha \circ h = g \circ \delta$, $(g - \alpha \circ k) \circ \delta = 0$. In other words, $g - \alpha \circ k$ is zero on $\operatorname{im} \delta = \ker \gamma$, and thus we have a map $\ell\colon M \to B$ with $g - \alpha \circ k = \ell \circ \gamma$. Then

$$\beta \circ \ell \circ \gamma = \beta \circ \bigl(g - \alpha \circ k\bigr) = \beta \circ g - (\beta \circ \alpha) \circ k = f \circ \gamma - 0 = f \circ \gamma,$$

so this shows $f = \beta \circ \ell$, i.e., the kernel of the map $\mathrm{Hom}(M, C) \to \mathrm{Ext}^1_R(M, A)$ is the image of $\mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C)$. $\square$

3. Show that $\mathrm{Ext}^1_R(M, A)$ is independent of the choice of P_1, P_2, and P_3, so that the notation $\mathrm{Ext}^1_R(M, A)$ makes sense. Ext^1_R is the simplest example of what is called a derived functor; there are many other examples in algebra.

Solution. Suppose given two choices

$$P_3 \xrightarrow{\varepsilon} P_2 \xrightarrow{\delta} P_1 \xrightarrow{\gamma} M \to 0.$$

and

$$P_3' \xrightarrow{\varepsilon'} P_2' \xrightarrow{\delta'} P_1' \xrightarrow{\gamma'} M \to 0.$$

for a "projective resolution" of M. We will first show that we can construct maps f_1, f_2, f_3, g_1, g_2, g_3 making the diagram

$$\begin{tikzcd}
P_3 \arrow[r, "\varepsilon"] \arrow[d, dotted, "f_3"] & P_2 \arrow[r, "\delta"] \arrow[d, dotted, "f_2"] & P_1 \arrow[r, "\gamma"] \arrow[d, dotted, "f_1"] & M \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\
P_3' \arrow[r, "\varepsilon'"] \arrow[d, dotted, "g_3"] & P_2' \arrow[r, "\delta'"] \arrow[d, dotted, "g_2"] & P_1' \arrow[r, "\gamma'"] \arrow[d, dotted, "g_1"] & M \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\
P_3 \arrow[r, "\varepsilon"] & P_2 \arrow[r, "\delta"] & P_1 \arrow[r, "\gamma"] & M \arrow[r] & 0
\end{tikzcd}$$

commute. We construct f_1 first, using the fact that P_1 is projective, which means that we can fill in the diagram

$$\begin{tikzcd} & P_1 \arrow[dl, dotted, "f_1"'] \arrow[d, "\gamma"] & \\ P_1' \arrow[r, "\gamma'"] & M \arrow[r] & 0. \end{tikzcd}$$

Then we construct f_2 and f_3 the same way, using projectivity of P_2 and P_3, respectively, and also construct the g's by the same method, with the P's and P''s interchanged. Then f_1, f_2, f_3, g_1, g_2, and g_3 give rise to a commuting diagram

$$\begin{tikzcd} \operatorname{Hom}(P_1, A) \arrow[r, "\delta^*"] & \operatorname{Hom}(P_2, A) \arrow[r, "\varepsilon^*"] & \operatorname{Hom}(P_3, A) \\ \operatorname{Hom}(P_1', A) \arrow[r, "\delta'^*"] \arrow[u, "{f_1}^*"] & \operatorname{Hom}(P_2', A) \arrow[r, "\varepsilon'^*"] \arrow[u, "{f_2}^*"] & \operatorname{Hom}(P_3', A) \arrow[u, "{f_3}^*"] \\ \operatorname{Hom}(P_1, A) \arrow[r, "\delta^*"] \arrow[u, "{g_1}^*"] & \operatorname{Hom}(P_2, A) \arrow[r, "\varepsilon^*"] \arrow[u, "{g_2}^*"] & \operatorname{Hom}(P_3, A), \arrow[u, "{g_3}^*"] \end{tikzcd}$$

which induces maps

$$\ker \varepsilon'^* / \operatorname{im} \delta'^* \underset{g^*}{\overset{f^*}{\rightleftarrows}} \ker \varepsilon^* / \operatorname{im} \delta^*.$$

We want to show these give isomorphisms. Since the P's and P''s play symmetrical roles, we only need to show that $f^* \circ g^* = \mathrm{id}$ on $\ker \varepsilon^* / \operatorname{im} \delta^*$. To see this, observe that $\gamma \circ (g_1 \circ f_1) = \gamma$, so that $\gamma \circ \big(g_1 \circ f_1 - \mathrm{id}_{P_1}\big) = 0$, i.e., $\operatorname{im}\big(g_1 \circ f_1 - \mathrm{id}_{P_1}\big) \subseteq \ker \gamma = \operatorname{im} \delta$. Since P_1 is projective, that implies there is a map $k_1 \colon P_1 \to P_2$ with $\delta \circ k_1 = g_1 \circ f_1 - \mathrm{id}_{P_1}$. Then $(g_1 \circ f_1 - \mathrm{id}_{P_1}) \circ \delta = \delta \circ (g_2 \circ f_2 - \mathrm{id}_{P_2}) = \delta \circ k_1 \circ \delta$, so $\delta \circ \big(g_2 \circ f_2 - \mathrm{id}_{P_2} - k_1 \circ \delta\big) = 0$ and $\operatorname{im}\big(g_2 \circ f_2 - \mathrm{id}_{P_2} - k_1 \circ \delta\big) \subseteq \ker \delta = \operatorname{im} \varepsilon$. Since P_2 is projective, that implies there is a map $k_2 \colon P_2 \to P_3$ with $\varepsilon \circ k_2 = g_2 \circ f_2 - \mathrm{id}_{P_2} - k_1 \circ \delta$.

Now suppose $\phi \colon P_2 \to A$ lies in $\ker \varepsilon^*$. Then

$$f^* \circ g^*(\phi) = \phi \circ g_2 \circ f_2 = \phi \circ \big(\varepsilon \circ k_2 + \mathrm{id}_{P_2} + k_1 \circ \delta\big) = \phi + \phi \circ \big(\varepsilon \circ k_2 + k_1 \circ \delta\big).$$

This represents the same class as ϕ, since $\phi \circ \varepsilon = 0$ and $\phi \circ k_1 \circ \delta$ lies in $\operatorname{im} \delta^*$. So $f^* \circ g^*$ induces the identity on $\ker \varepsilon^* / \operatorname{im} \delta^*$, and by symmetry, similarly with $g^* \circ f^*$. $\square$

4. Let $R = \mathbb{Z}$, $M = \mathbb{Z}/k$, and choose $P_1 = P_2 = \mathbb{Z}$. Show that $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/k, A)$ is just A/kA, and check the exact sequence

$$0 \to \operatorname{Hom}(\mathbb{Z}/k, A) \xrightarrow{\alpha_*} \operatorname{Hom}(\mathbb{Z}/k, B) \xrightarrow{\beta_*} \operatorname{Hom}(\mathbb{Z}/k, C) \to \operatorname{Ext}^1_R(\mathbb{Z}/k, A)$$

directly.

Solution. Take $P_3 = 0$ and $P_2 \xrightarrow{\delta} P_1$ to be $\mathbb{Z} \xrightarrow{k} \mathbb{Z}$. Since $\operatorname{Hom}(\mathbb{Z}, A)$ is naturally isomorphic to A, $\operatorname{Hom}(P_1, A) \xrightarrow{\delta^*} \operatorname{Hom}(P_2, A)$ is just $A \xrightarrow{k} A$, whose kernel is the k-torsion in A (usually denoted ${}_kA$), and whose cokernel is A/kA. So $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/k, A) = A/kA$. Also note that $\operatorname{Hom}(\mathbb{Z}/k, A) = {}_kA$, since a homomorphism $\mathbb{Z}/k \to A$ is uniquely determined by the image of the coset of 1, which can be any k-torsion class in A. So the exact sequence simplifies in this case to

$$0 \to {}_kA \xrightarrow{\alpha_*} {}_kB \xrightarrow{\beta_*} {}_kC \to A/kA.$$

To check the exact sequence, we only need to see how to define a map ${}_kC \to A/kA$ and check exactness at ${}_kC$. Given a k-torsion element x in C, lift it to an element $y \in B$. Then ky is not necessarily 0, but $\beta(ky) = k\beta(y) = kx = 0$. So $ky \in \ker\beta = \operatorname{im}\alpha$ and $ky = \alpha(z)$ for a unique $z \in A$. Note that z depended on a choice, as we had to choose y mapping to x, and we are free to modify y by any element of $\ker\beta = \operatorname{im}\alpha$. In other words, y is only well determined up to an element of A, so ky is only well determined up to an element of kA,and z is well determined in A/kA. So we get a map ${}_kC \to A/kA$. The kernel of this map consists of elements of C that lift to k-torsion elements of B, or in other words, precisely the image of ${}_kB \xrightarrow{\beta_*} {}_kC$. $\square$