MATH 601: Abstract Algebra II
3rd Homework Solutions
Hungerford, Exercise V.3.6

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assignment due Monday, February 19, 2001

(a) We are given $f(x)$, $g(x)$ relatively prime in $K[x]$, such that $\frac{f(x)}{g(x)} \notin K$. In particular, $f(x)$ and $g(x)$ each have degree at least 1. Now x is a root of the polynomial $\varphi(y) = \left(\frac{f(x)}{g(x)}\right) g(y) - f(y) \in K\left(\frac{f(x)}{g(x)}\right)[y]$, since $\varphi(x) = \left(\frac{f(x)}{g(x)}\right) g(x) - f(x) = 0$, so x is algebraic over $\frac{f(x)}{g(x)}$. Furthermore, as a polynomial in y, φ has as its degree $\max(\deg f, \deg g)$.

Now $\left(\frac{f(x)}{g(x)}\right)$ is transcendental over K, since if it were a root of a polynomial $h(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 \in K[z]$, we could clear denominators in

$$\left(\frac{f(x)}{g(x)}\right)^n + a_{n-1}\left(\frac{f(x)}{g(x)}\right)^{n-1} + \cdots + a_0 = 0$$

by multiplying by $g(x)^n$ to get a polynomial equation with coefficients in K,

$$f(x)^n + a_{n-1}f(x)^{n-1}g(x) + \cdots + a_0\, g(x)^n = 0,$$

satisfied by x, contradicting the fact that x is transcendental over K. So let $z = \frac{f(x)}{g(x)}$; we have $K\left(\frac{f(x)}{g(x)}\right) = K(z)$, the field of rational functions in the transcendental element z. We want to show $\varphi(y) = zg(y) - f(y)$ is irreducible in $K(z)[y]$. Note that in fact $\varphi(y) \in K[z][y]$, and it's primitive as a polynomial over $K[z]$, since its coefficients are the coefficients of g, multiplied by z, and the coefficients of f, so the only common factors of all the coefficients are non-zero elements of the ground field K, which are units in $K[z]$. So by Gauss's Lemma, φ can have a nontrivial factorization only if it factors in $K[z][y] = K[y, z]$. Since φ has degree 1 in z, if it had such a factorization $\varphi(y) = \varphi_1(y)\varphi_2(y)$, then one could assume φ_1 had degree 1 in z and φ_2 had degree 0 in z, i.e., were an element of $K[y]$. But then we'd have $\varphi_2(y)$ dividing $zg(y) - f(y)$, hence dividing both $g(y)$ and $f(y)$, which is impossible unless φ_2 is a constant, since f and g were assumed relatively prime. Thus φ is irreducible and x has degree precisely $\max(\deg f, \deg g)$ over $K(z)$.

(b) Assume $K \subset\neq E \subseteq K(x)$ with E a field. Then E contains some element $\frac{f(x)}{g(x)}$ of $K(x)$ not in K. Hence we can apply part (a) to conclude that $[K(x) : K\left(\frac{f(x)}{g(x)}\right)] < \infty$, and so $[K(x) : E] < \infty$, since $K\left(\frac{f(x)}{g(x)}\right) \subseteq E$.

(c) We saw above that any element $z = \frac{f(x)}{g(x)}$ of $K(x)$ which is not in K is transcendental over K. So $K(z) \cong K(x)$ and there is a monomorphism $\sigma\colon K(x) \to K(z) \subseteq K(x)$ which is the identity on K and sends $x \mapsto z$. For any rational function h, this monomorphism sends $h(x) \mapsto h(y)$. Note that σ is a K-automorphism of $K(x)$ if and only if it is surjective. In this case, we have $K(z) = K(x)$. Since, by part (b), $[K(x) : K(z)] = \max(\deg f, \deg g)$, we see σ is an automorphism of $K(x)$ if and only of $\max(\deg f, \deg g) = 1$.

(d) By part (c), $\mathrm{Aut}_K\, K(x)$ can be identified precisely with the maps $x \mapsto \frac{f(x)}{g(x)}$, where $\max(\deg f, \deg g) = 1$. Thus these are the maps

$$x \mapsto \frac{ax+b}{cx+d}$$

where a, b, c, $d \in K$, a and c are not both 0, $b \neq 0$ if $a = 0$ and $d \neq 0$ is $c = 0$, and $ax + b$ and $cx + d$ are not multiples of each other. The criteria on a, b, c, d translate into saying that the vectors (a, b) and (c, d) in K^2 are linearly independent, or that

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0.$$