MATH 601: Abstract Algebra II
10th Homework
Partial Solutions

Jonathan Rosenberg
(with thanks to Ben Howard for a TEX file of his solution)

assignment due Wednesday, May 9, 2001

1 Additional exercises (the spectrum of a ring)

5. Let R be a commutative ring (with unit), $X = \operatorname{Spec} R$. Suppose X is disconnected, i.e., that X is the union of two disjoint closed subsets. Deduce that R contains an idempotent e (an element with $e^2 = e$) other than 0 and 1. (Hint: Chinese Remainder Theorem.) Also prove the converse (this is easier).
Solution. Suppose $X = \operatorname{Spec}(R) = A \amalg B$ with A and B disjoint closed sets. By definition of what it means to be a closed set, there are two ideals I and J such that $A = \{P \in X : I \subseteq P\}$ and $B = \{P \in X : J \subseteq P\}$. Since A and B are disjoint, no prime ideal contains both I and J. Therefore, $I + J = R$. Since $A \cup B = X$, every prime ideal of R contains $I \cap J$, so the nilradical N of R (the intersection of all the prime ideals) contains $I \cap J$. Hence in $\overline{R} = R/N$, the images $\overline{I}$ and $\overline{J}$ of I and J satisfy $\overline{I} + \overline{J} = \overline{R}$, $\overline{I} \cap \overline{J} = 0$. By the Chinese Remainder Theorem, $\overline{R} \cong (\overline{R}/\overline{I}) \times (\overline{R}/\overline{J})$. So the element $(0, 1) \in (\overline{R}/\overline{I}) \times (\overline{R}/\overline{J})$ pulls back to an element x of I that is idempotent mod N, or such that $x(1-x)$ is nilpotent. Choose $e \in \{x \in I : 1 - x \in J\}$ such that $\min\{n \geq 1 : e^n(1-e)^n = 0\}$ is minimal. We claim that e is idempotent. Let $n = \min\{n \geq 1 : e^n(1-e)^n\}$. Now e is idempotent if and only if $n = 1$.

Motivation: Since $e^n(1-e)^n = 0$, we can think of e as an approximation to a root of the polynomial $x - x^2$. One method to get better approximations is Newton's method from Calculus. Of course, we need to show that this method makes sense for our situation. After all, we are not dealing with a continuous real valued function! Using Newton's method, the next approximation after e is $f = e - \frac{e-e^2}{1-2e}$. We must first show that $1 - 2e$ is invertible (a unit), and next verify that the essential property of e (that $e \in I$ and $1 - e \in J$) is retained by f.

We show $1 - 2e$ is a unit. We have

$$1 - 4e(1-e) = ((1-e) + e)^2 - 4e(1-e) = ((1-e) - e)^2 = (1-2e)^2.$$

Since 1 is a unit and $4e(1-e)$ is nilpotent, we know that the difference is a unit. Thus $(1-2e)^2$ is a unit and consequently $1 - 2e$ is a unit.

We have that

$$f = e - \frac{e - e^2}{1 - 2e} = \frac{e(1-2e) - e + e^2}{1 - 2e} = \frac{-e^2}{1 - 2e} \quad \text{and} \quad 1 - f = 1 + \frac{e^2}{1 - 2e} = \frac{(1-e)^2}{1 - 2e}.$$

Since $e \in I$ we have $f \in I$, and since $1 - e \in J$ we have that $1 - f \in J$.

Suppose n is even. Then $n = 2k$ for some positive integer k. So,

$$f^k(1-f)^k = (-1)^k \frac{e^{2k}(1-e)^{2k}}{(1-2e)^{2k}} = 0.$$

But $k < n$ and this contradicts the minimality of n.

Suppose n is odd. Then $n = 2k - 1$ for some integer $k \geq 1$. Since $e^n(1-e)^n = 0$ we also have that $e^{n+1}(1-e)^{n+1} = e^{2k}(1-e)^{2k} = 0$. So again,

$$f^k(1-f)^k = \frac{e^{2k}(1-e)^{2k}}{(1-2e)^{2k}} = 0.$$

Of course, $n = 2k - 1$ implies that $k \leq n$, and from the minimality of n we have that $n \leq k$. Therefore $n = k$. But also $n = 2k - 1$ so $n = k = 1$.

Therefore, $e(1-e) = 0$, and so e is an idempotent element. That e is nontrivial follows from the fact that neither I nor J can contain 1. □