MATH 601: Abstract Algebra II
6th Homework
Partial Solutions

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assignment due Wednesday, April 18, 2001

1. Find representatives for all the equivalence classes of irreducible (complex) representations of $G = Q_8$.
Solution. Note that G is generated by i and j, so its commutator subgroup is generated by $iji^{-1}j^{-1} = j^{-2} = -1$, a central element of order 2. So the maximal abelian quotient of G has order 4, and in this quotient, the images of i and j each have order 2. So G has 4 one-dimensional representations, given by $i \mapsto \pm 1$ and $j \mapsto \pm 1$. In addition, G obviously has a representation π by left multiplication on $\mathbb{H}$, which is a 2-dimensional vector space over $\mathbb{C}$ (acting on the *right*) with basis $e_1 = 1$ and $e_2 = j$. Note that $\pi(i)$ sends $1 = e_1$ to $i \cdot e_1$ and $j = e_2$ to $i \cdot j = -j \cdot i = -ie_2$, while $\pi(j)$ sends $1 = e_1$ to $j = e_2$ and $j = e_2$ to $-1 = -e_1$. So the representation π is determined by

$$i \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad k \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

with respect to this basis. The character χ_π is thus given by

$$\chi_\pi(1) = 2, \quad \chi_\pi(-1) = -2, \quad \chi_\pi(\pm i) = \chi_\pi(\pm j) = \chi_\pi(\pm k) = 0.$$

The squared L^2-norm of χ_π is $(4 + 4)/|G| = 1$, so π is irreducible, and we've found all the irreducible representations since G only has 5 conjugacy classes ($\{1\}$, $\{-1\}$, $\{\pm i\}$, $\{\pm j\}$, and $\{\pm k\}$). $\square$

2. Let $G = A_4$, the alternating group on 4 letters, which has order 12. Find representatives for all the equivalence classes of irreducible (complex) representations of G, and compute their characters.
Solution. Note that G has a normal (in fact, characteristic) Sylow 2-subgroup

$$H = \{1, (12)(34), (13)(24), (14)(23)\},$$

and G/H is cyclic of order 3. So G has 3 one-dimensional representations with H mapping to 1: the trivial representation (everything goes to 1), a representation with $(123) \mapsto e^{2\pi i/3}$, and a representation with $(123) \mapsto e^{-2\pi i/3}$. One can get another irreducible 3-dimensional representation π by inducing from H up to G the one-dimensional representation ρ of H with $(12)(34) \mapsto -1$, $(13)(24) \mapsto 1$, and $(14)(23) \mapsto -1$. This representation has dimension $[G : H] = 3$, so assuming it's irreducible (which we'll check in a minute), we've accounted for all the irreducible representations, since $3 \cdot 1^2 + 3^2 = 12 = |G|$. To check that π is irreducible, we can use the Frobenius reciprocity theorem: $\dim \mathrm{Hom}_G(\pi, \pi) = \dim \mathrm{Hom}_H(\rho, \pi|_H)$, and since H is normal, $\pi|_H = \mathbb{C}G \otimes_{\mathbb{C}H} \mathbb{C}_\rho$ as an H-module via the action from the left is simply $\rho \oplus (123) \cdot \rho \oplus (123)^2 \cdot \rho$ (since

1, (123), and $(123)^2$ are representatives for the cosets of H in G). But ρ, $(123)\cdot\rho$, and $(123)^2\cdot\rho$ are the 3 non-trivial characters of H, so $\dim \mathrm{Hom}_H(\rho, \pi|_H) = 1$ and π is irreducible. Not only that, but we've seen that π restricted to H looks like the complement of the trivial representation in the regular representation of H, while π restricted to a Sylow 3-subgroup is just the regular representation. So now we can compute the character χ_π of π. On H, it's the character of the regular representation, which is 4 at 1 and 0 elsewhere, minus the character of the trivial representation, which is everywhere 1. So

$$\chi_\pi(1) = 3, \quad \chi_\pi = -1 \text{ on } H \smallsetminus \{1\}, \quad \chi_\pi = 0 \text{ on 3-cycles.}$$

Incidentally, we see from this that the squared L^2-norm of χ_π is $(3^2 + 3\cdot(-1)^2)/|G| = 1$, which once again confirms that π is irreducible. Another way to realize π is as follows: it acts on the vector space

$$V = \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$$

with G acting by permuting the coordinates. We can choose as a basis for V the vectors $e_1 = (1,0,0,-1)$, $e_2 = (0,1,0,-1)$, and $e_3 = (0,0,1,-1)$. Then (123) permutes e_1, e_2, and e_3 cyclically. On the other hand, (12)(34) sends e_1 to $(0,1,-1,0) = e_2 - e_3$, e_2 to $(1,0,-1,0) = e_1 - e_3$, and e_3 to $-e_3$, and (13)(24) sends e_1 to $(0,-1,1,0) = e_3 - e_2$, e_2 to $-e_2$, and e_3 to $(1,-1,0,0) = e_1 - e_2$. So in terms of explicit matrices, the representation π can be realized by

$$(123) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad (12)(34) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}, \quad (13)(24) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix},$$

since G is generated by (123), (12)(34) and (13)(24). □

3. Let $G = S_4$, the symmetric group on 4 letters, which has order 24. Find representatives for all the equivalence classes of irreducible (complex) representations of G, and compute their characters. Using your answer to #2, determine how the representations restrict to A_4.
Solution. We have two obvious 1-dimensional representations of G, the trivial representation 1 (sending everything to 1) and the sign representation sgn (sending A_4 to 1, odd permutations to -1). Since $\mathrm{sgn}^2 = 1$, tensoring with sgn is a permutation of $\widehat{G}$ of order 2. Also we have the 3-dimensional representation π of G on

$$V = \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$$

with G acting by permuting the coordinates. We saw in #2 that the restriction of this representation to A_4 is irreducible; hence *a fortiori* π is irreducible. We already computed the character of π on even permutations. With the same basis as in #2, (12) interchanges e_1 and e_2 and fixes e_3, so χ_π must be 1 on 2-cycles. Thus $\chi_{\pi\otimes\mathrm{sgn}}$ must be -1 on 2-cycles, so π and $\pi\otimes\mathrm{sgn}$ must be inequivalent. We can also compute that the 4-cycle (1234) sends e_1 to $(-1,1,0,0) = e_2 - e_1$, e_2 to $(-1,0,1,0) = e_3 - e_1$, and e_3 to $(-1,0,0,1) = -e_1$. So χ_π must be -1 on 4-cycles, while $\chi_{\pi\otimes\mathrm{sgn}}$ must be $+1$ on 4-cycles.

Now we've found two 1-dimensional representations and two irreducible 3-dimensional representations. Since $2\cdot 1^2 + 2\cdot 3^2 = 20$, we're still missing something. But there is another irreducible 2-dimensional representation. If H is as in #2, then H is normal in G and $G/H \cong S_3 \cong D_3$, which has a unique irreducible 2-dimensional representation σ. This representation can be lifted to an irreducible 2-dimensional representation of G in which H acts trivially. We can realize the representation by

$$(123) \mapsto \begin{pmatrix} e^{2\pi i/3} & 0 \\ 0 & e^{-2\pi i/3} \end{pmatrix}, \quad (12) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since σ is trivial on H, $\chi_\sigma = 2$ on the identity and on the conjugacy class of $(12)(34)$, and $\chi_\sigma = 0$ on 2-cycles, $\chi_\sigma = e^{2\pi i/3} + e^{-2\pi i/3} = -1$ on 3-cycles. Finally, since $(1234) = (13)(12)(34)$, χ_σ is the same on 4-cycles as it is on 2-cycles. $\square$