MATH 603, SPRING 2011
HOMEWORK ASSIGNMENT #10 ON DIMENSION THEORY: SOLUTIONS

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(1) Is there a purely topological proof of the fact that Noetherian rings have the descending chain condition on prime ideals? In other words, answer the following questions:

(a) What property of a Noetherian topological space X corresponds to the descending chain condition on prime ideals of a Noetherian ring R when $X = \operatorname{Spec} R$?

Solution. If $X = \operatorname{Spec} R$, then prime ideals of R are just points of X. For prime ideals $\mathfrak{p}$ and $\mathfrak{p}'$, $\mathfrak{p} \subseteq \mathfrak{p}'$ if and only if $\mathfrak{p}'$ lies in the closure of $\mathfrak{p}$ (as points in the topological space $\operatorname{Spec} R$). So the DCC on prime ideals corresponds to the following: given a sequence of points $\{x_j\}$ in X with x_j in the closure of x_k whenever $j \leq k$, the sequence must be eventually constant. □

(b) Does every Noetherian topological space X have the above property? Give a proof or a counterexample.

Solution. No, this property is not automatic. Here is a counterexample. Let $X = \mathbb{N}$, the non-negative integers, with the non-empty open sets consisting of the intervals $[n, \infty)$, $n \in \mathbb{N}$. (Thus the closed sets, aside from $\emptyset$ and $\mathbb{N}$ itself, are the intervals $\{0, 1, \cdots, n\}$, $n \in \mathbb{N}$.) The sequence $\{0, 1, 2, \cdots\}$ has the property that each point is in the closure of the points that follow it, and this sequence is not eventually constant. On the other hand, X is Noetherian by the characterization that appeared in a previous homework: every open subspace is quasi-compact. Indeed, if $U \subsetneq X$ is open and non-empty, then $U = [n, \infty)$ for some n, so U is homeomorphic to X itself. So it's enough to show that X is quasi-compact. But given any open covering $\mathcal{U}$ of X, 0 must lie in one set of the covering, and this set is necessarily equal to X itself, so $\mathcal{U}$ has a finite (in fact a singleton) subcover. □

(2) Compute the Hilbert function of the graded ring $R = K[x, y, z]/(x^2z - y^3 - yz^2)$, where K is a field and x, y, z each have degree 1.

Solution. Observe that $f = x^2z - y^3 - yz^2$ is a homogeneous polynomial of degree 3. So if $S = K[x, y, z]$ is the polynomial ring, we have short exact sequences

$$0 \to S_n \xrightarrow{f} S_{n+3} \to R_{n+3} \to 0.$$

Thus $H_S(n + 3) = H_R(n + 3) + H_S(n)$, and multiplying by t^{n+3} and summing, we get $(1-t^3)P_S(t) = P_R(t)$. Since $P_S(t) = (1-t)^{-3}$, $P_R(t) = (1-t^3)(1-t)^{-3} = (1+t+t^2)(1-t)^{-2}$, from which we can read off the Hilbert function:

$$H_R(n) = \text{coefficient of } t^n \text{ in } (1 + t + t^2)(1 - t)^{-2} = (n + 1) + n + (n - 1) = 3n, \quad n > 0.$$

Of course, $H_R(0) = 1$, so the case $n = 0$ is exceptional. Note that the proof makes no use at all of the specific form of the polynomial f, other than the fact that it's homogeneous of degree 3. □

(3) Show that a regular Noetherian local ring R of dimension d is integrally closed.
Solution. Recall that we proved in class that R must be an integral domain. The proof is by induction on d. If $d = 0$, R is a field and this is obvious. So suppose $d > 0$ and the result is known for smaller values of d. Let $\mathfrak{m}$ be the maximal ideal of R, let K be the field of fractions of R, and let $\mathfrak{p} \subsetneq \mathfrak{m}$ be a prime ideal of height $d-1$, generated by $d-1$ elements of a system of parameters for R. Then $R_{\mathfrak{p}}$ is a regular local ring of dimension $d-1$, also with field of fractions K, so $R_{\mathfrak{p}}$ is integrally closed in K. Thus if an element of K is integral over R, it must lie in $R_{\mathfrak{p}}$, so we can write it in the form $x^{-n}y$, where $y \in R$ and $x \in \mathfrak{m}$, $x \notin \mathfrak{p}$, $x \notin \mathfrak{m}^2$. ($R_{\mathfrak{p}}$ is obtained from R by inverting non-units, i.e., elements of $\mathfrak{m}$, that don't lie in $\mathfrak{p}$.) But now x and $\mathfrak{p}$ must span $\mathfrak{m}$ modulo $\mathfrak{m}^2$, so (x) must be prime of height 1. ($R/(x)$ is a local ring of dimension $d-1$ with a maximal ideal generated modulo its square by $d-1$ elements.) Consider $R_{(x)}$, R localized at (x). This is a local ring with the same field of fractions K with a principal maximal ideal generated by x. So this is a DVR and is integrally closed. Thus $x^{-n}y$ lies in $R_{(x)}$, which can only happen if it lies in R. □