MATH 603, SPRING 2011
HOMEWORK ASSIGNMENT #7 ON NOETHERIAN RINGS AND SPACES
PARTIAL SOLUTIONS

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2. Since R is Noetherian, the zero-ideal (0) in R has a minimal primary decomposition $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$, where $\mathfrak{q}_j$ is $\mathfrak{p}_j$-primary and the $\mathfrak{p}_j$'s are the prime ideals associated to (0). By a theorem proved in class, each $\mathfrak{p}_j$ is of the form $(0 : x_j) = \operatorname{Ann} x_j$ for some $x_j \in R$, and thus consists of zero-divisors. In fact, the set of zero-divisors in R is precisely the union of the $\mathfrak{p}_j$'s, by Proposition 4.17 in A-M. Thus S is the intersection of the complements of the $\mathfrak{p}_j$'s.

Now the prime ideals of $S^{-1}R$ are precisely the $S^{-1}\mathfrak{p}$, with $\mathfrak{p}$ a prime ideal of R not meeting S. Not meeting S means $\mathfrak{p} \subseteq \bigcup_j \mathfrak{p}_j$. By the "Prime avoidance" theorem proved in class, that means $\mathfrak{p} \subseteq \mathfrak{p}_j$ for some j. If $S^{-1}\mathfrak{p}$ is maximal, then in fact $\mathfrak{p}$ must be equal to $\mathfrak{p}_j$ for some j, and so there are only finitely many maximal ideals in $S^{-1}R$.

If R is not Noetherian, this argument falls apart. In fact, if I is an infinite set and if $R = \prod_{i\in I} \mathbb{F}_2$, then R is a Boolean ring (each element is its own square, since this holds in $\mathbb{F}_2$ and passes to products). Furthermore, the only element of S is 1, since an element of R that is not the identity must have a coordinate 0 at some point $i \in I$, and then is a zero-divisor (its product with the element e_i that is 1 at this value of i and 0 everywhere else is 0). So $S^{-1}R = R$, which is not Noetherian. Indeed, if $i_1, i_2, \cdots$ is an infinite sequence in I, then

$$(0) \subsetneq (e_{i_1}) \subsetneq (e_{i_1}, e_{i_2}) \subsetneq (e_{i_1}, e_{i_2}, e_{i_3}) \subsetneq \cdots$$

in an infinite non-terminating ascending chain of ideals. The kernel of evaluation at $i \in I$, $p_i \colon \prod_{i\in I} \mathbb{F}_2 \to \mathbb{F}_2$, is a maximal ideal $\mathfrak{p}_i$ that contains $1 - e_i$ but not $1 - e_j$ for any $j \neq i$, so R has infinitely many maximal ideals.