MATH 608K (Algebraic K-Theory)
Homework Assignment #4
K_2 and Symbols
Partial Solutions

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1. Do Exercise 4.4.28 in the book. In other words, show the following about the quaternion algebras $A_F(a,b)$ (with basis 1, x, y, xy, and relations $x^2 = a \in F^\times$, $y^2 = b \in F^\times$, and $xy = -yx$) over a field F of characteristic $\neq 2$:

   (a) Show that $A_F(a,b) \cong A_F(b,a)$ is anti-isomorphic to itself, and thus defines an element of order 2 in the Brauer group of F. (What this means in concrete terms is that $A_F(a,b) \otimes_F A_F(a,b) \cong M_4(F)$.)

   Solution. The usual anti-involution, $x \mapsto \bar{x} = -x$, $y \mapsto \bar{y} = -y$, $xy \mapsto \overline{xy} = -xy$, gives an isomorphism $A_F(a,b) \to A_F(a,b)^{\text{op}}$. But for any d-dimensional central simple algebra over F, $A \otimes_F A^{\text{op}} \cong M_d(F)$.

   (b) Show that $A_F(a,b) \otimes_F A_F(a,c) \cong M_2(A_F(a,bc))$, for $a,b,c \in F^\times$, that $A_F(a,-a) \cong M_2(F)$, and that $A_F(a,1-a) \cong M_2(F)$ for $a \neq 1$.

   Solution. For the last two statements it suffices to prove the corresponding fact for the Hilbert symbol, since $A_F(a,b) \cong M_2(F)$ if and only if $(a,b)_F = 1$. But vanishing of $(a,-a)_F$ and of $(a,1-a)_F$ is trivial, since $a \cdot 1^2 - a \cdot 1^2 = 0^2$ and $a \cdot 1^2 + (1-a) \cdot 1^2 = 1^2$.

   The hard part of course is the first statement. One can prove it as follows. Let x and y be anti-commuting generators of $A_F(a,b)$, with $x^2 = a$, $y^2 = b$, and let u and v be anti-commuting generators of $A_F(a,c)$, with $u^2 = a$, $v^2 = c$. Then $x \otimes u$ and $x \otimes uv$ anti-commute in $A_F(a,b) \otimes_F A_F(a,c)$, and have squares $x^2 \otimes u^2 = a \otimes a = a^2(1 \otimes 1)$ and $x^2 \otimes uvuv = -x^2 \otimes u^2v^2 = -a \otimes ac = -a^2c(1 \otimes 1)$, so they generate a copy of $A_F(a^2, -a^2c)$ inside $A_F(a,b) \otimes_F A_F(a,c)$. But since a^2 is clearly a perfect square in $F^\times$, the Hilbert symbols $(a^2, -a^2c)_F$ is $+1$ and thus $A_F(a^2, -a^2c) \cong M_2(F)$. At the same time, $x \otimes 1$ and $y \otimes v$ anti-commute in $A_F(a,b) \otimes_F A_F(a,c)$, and have squares $x^2 \otimes 1 = a \otimes 1 = a(1 \otimes 1)$ and $y^2 \otimes v^2 = bc(1 \otimes 1)$, so they generate a copy of $A_F(a,bc)$ inside $A_F(a,b) \otimes_F A_F(a,c)$. Since $x \otimes 1$ and $y \otimes v$ each commute with $x \otimes u$ and $x \otimes uv$, the copies of $M_2(F)$ and of $A_F(a,bc)$ inside $A_F(a,b) \otimes_F A_F(a,c)$ commute with each other. So we get a homomorphism $A_F(a,bc) \otimes M_2(F) \to A_F(a,b) \otimes_F A_F(a,c)$. Since all algebras are simple and the dimensions agree, this must be an isomorphism.

   (c) Show in this way that one gets a homomorphism $\{a,b\} \mapsto [A_F(a,b)]$ from $K_2(F)$ to a 2-torsion subgroup of the Brauer group of F, generated by stable isomorphism classes of quaternion algebras over F.

2. See if you can give an explicit calculation of $K_2(\mathbb{Q}(i))$, following the same outline as for $K_2(\mathbb{Q})$, except

that remember you want to replace ordinary primes by primes in the Euclidean ring of Gaussian integers. These primes, modulo multiplication by units, are: $1+i$, which up to a unit is the same as $1-i$, $a \pm bi$ with $a > b > 0$, $a^2+b^2 = p$, when p is a prime $\equiv 1\,(4)$, and ordinary primes $p \equiv 3\,(4)$.

Solution. By the localization sequence, there is an exact sequence

$$K_2(\mathbb{Z}[i]) \to K_2(\mathbb{Q}(i)) \xrightarrow{\partial} \bigoplus_{\mathfrak{p}} K_1(\mathbb{Z}[i]/\mathfrak{p}) \to K_1(\mathbb{Z}[i]) \to K_1(\mathbb{Q}(i)).$$

Here the sum is over all maximal ideals $\mathfrak{p}$ of $\mathbb{Z}[i]$. There is one of these, $(1+i)$, for which the quotient $\mathbb{Z}[i]/\mathfrak{p}$ is isomorphic to $\mathbb{F}_2$, and over odd ordinary primes p, there are either one or two possibilities for $\mathfrak{p}$. Since $\mathbb{Z}[i]$ is a Euclidean ring, $K_1(\mathbb{Z}[i]) \cong \mathbb{Z}[i]^\times \cong \{\pm 1, \pm i\}$, which maps injectively into $K_1(\mathbb{Q}(i)) \cong \mathbb{Q}(i))^\times$. Thus the map ∂ is surjective. But it is also known that $K_2(\mathbb{Z}[i])$ is generated by Steinberg symbols. Since $\mathbb{Z}[i]^\times \cong \{\pm 1, \pm i\}$ is cyclic with generator $i = \sqrt{-1}$, $K_2(\mathbb{Z}[i])$ is thus the cyclic group generated by $\{i, i\}$. But $\{i,i\}^3 = \{i, i^3\} = \{i, -i\} = 1$, while $\{i,i\}^4 = \{i, i^4\} = \{i, 1\} = 1$, so $\{i,i\}^3 = \{i,i\}^4 = 1$, $\{i,i\} = 1$, and $K_2(\mathbb{Z}[i])$ vanishes. Hence we obtain

$$\begin{aligned} K_2(\mathbb{Q}(i)) \xrightarrow[\cong]{\partial} \bigoplus_{\mathfrak{p}} K_1(\mathbb{Z}[i]/\mathfrak{p}) &\cong \bigoplus_{p\equiv 1 \bmod 4} \left(\mathbb{F}_p^\times\right)^2 \oplus \bigoplus_{p \equiv 3 \bmod 4} \mathbb{F}_{p^2}^\times \\ &\cong \bigoplus_{p\equiv 1 \bmod 4} \left(C_{p-1} \times C_{p-1}\right) \oplus \bigoplus_{p\equiv 3 \bmod 4} C_{p^2-1}. \end{aligned}$$

The explanation of the calculation is as follows. $\mathbb{F}_2^\times$ is trivial anyway. Over $p \equiv 1 \bmod 4$, there are two primes $a \pm bi$ with $a^2 + b^2 = 1$, $a > b > 0$, with $\mathbb{Z}[i]/(a \pm bi) \cong \mathbb{F}_p$, while $p \equiv 3 \bmod 4$ remains prime and gives a quotient $\mathbb{Z}[i]/(p) \cong \mathbb{F}_p(i)$ isomorphic to $\mathbb{F}_{p^2}$.

One can also arrive at the same result, without having to quote results about $K_2(\mathbb{Z}[i])$, by mimicking the proof used for studying $K_2(\mathbb{Q})$. Let A_m be the subgroup of $K_2(\mathbb{Q}(i))$ generated by symbols $\{z, w\}$, with z and w non-zero Gaussian integers of norm $\le m$. Then obviously $K_2(\mathbb{Q}(i)) = \varinjlim A_m$. We see that A_1 is generated by $\{i, i\}$, which vanishes as above. So A_2 is generated by $\{1+i, -i\} = 1$ (since $(1+i) + (-i) = 1$) and by $\{1+i, 1+i\}$. But the latter is $\{1+i, -1-i\}\{1+i, -1\} = \{1+i, (-i)^2\} = \{1+i, -i\}^2 = 1$, so A_2 vanishes also. Then for $m > 2$, by factorization and bilinearity, it is clear that $A_m = A_{m-1}$ unless m is an odd prime $p \equiv 1 \bmod 4$ or the square of an odd prime $p \equiv 3 \bmod 4$. So the idea is to set up a homomorphism $\left(\mathbb{F}_p^\times\right)^2 \to A_p/A_{p-1}$ if p is a prime $p \equiv 1 \bmod 4$ and a homomorphism $\mathbb{F}_{p^2}^\times \to A_{p^2}/A_{p^2-1}$ if p is a prime $p \equiv 3 \bmod 4$. Let's do the latter, say. We have $\mathbb{Z}[i]/(p) \cong (\mathbb{Z}/p) + i(\mathbb{Z}/p) \cong \mathbb{F}_p[\sqrt{-1}] \cong \mathbb{F}_{p^2}$, so an element of $\mathbb{F}_{p^2}^\times$ is represented by $z = x + iy$ with x and y integers, $|x|$, $|y| \le \frac{p-1}{2}$, x and y not both 0. The norm of this element is $x^2 + y^2 < \frac{2(p-1)^2}{4} < p^2$. Thus $\{z, p\}$ is an element of A_p. Map the residue class of z in $\mathbb{F}_{p^2}^\times$ to the class of $\{z, p\}$ modulo A_{p^2-1}. To show this is a homomorphism, suppose $zw = up + r$ with u and r having norm less than p^2. (This is possible since $\mathbb{Z}[i]$ is a Euclidean ring.) Then we have

$$1 = \left\{1 - \frac{up}{zw}, \frac{up}{zw}\right\} = \left\{\frac{r}{zw}, \frac{up}{zw}\right\},$$

which by bilinearity is the same as

$$\{r, up\}\{zw, up\}^{-1}\{r, zw\}^{-1}\{zw, zw\}.$$

Since $\{r, up\} = \{r, u\}\{r, p\}$ with $\{r, u\} \in A_{p^2-1}$, and similarly with the other terms, this identity implies that $\{z, p\}\{w, p\} = \{r, p\}$ modulo A_{p^2-1}, and thus we have a homomorphism. Then one shows that it's injective and surjective. The other case is similar.