MATH 608K (Algebraic K-Theory)
Homework Assignment #1, Fall, 2005

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1. Let $R = C^{\mathbb{R}}(S^1)$, the continuous real-valued functions on the circle. We identify S^1 with the unit circle in the complex plane. It is sometimes convenient to think of S^1 as the result of gluing together the two endpoints of the interval $[0, 2\pi]$, via the map $\theta \mapsto e^{i\theta}$.
   (a) Show that every finitely generated projective R-module P has a well-defined rank $n \in \mathbb{N}$. (Hint: Each point $z \in S^1$ defines an evaluation homomorphism $\phi_z : R \twoheadrightarrow \mathbb{R}$, whose kernel is a maximal ideal. So for each $z \in S^1$, $P \otimes_{\phi_z} \mathbb{R}$ is a finite-dimensional vector space over $\mathbb{R}$. Argue that the dimension n of this vector space must vary continuously with z, hence must be constant since S^1 is connected.)
   (b) Show that if P (a finitely generated projective R-module) has rank 1, then either $P \cong R$ or else $P \cong M$, where
   $$M = \{f \in C^{\mathbb{R}}(S^1) : f(z) = -f(-z)\},$$
   viewed as an R-module by identifying R with
   $$\{f \in C^{\mathbb{R}}(S^1) : f(z) = f(-z)\},$$
   by means of the covering map $z \mapsto z^2$, and letting R act on M by multiplication. (An odd function multiplied by an even function is odd.) In this part of the problem, it helps to think of S^1 as a quotient of $[0, 2\pi]$, and to use the fact (which you may assume) that every finitely generated projective $C^{\mathbb{R}}([0, 2\pi])$-module is free.
   (c) Show that with M as in (b), $M \oplus R^{n-1} \not\cong R^n$ as R-modules. (Hint: Every $f \in M$ vanishes somewhere by the Intermediate Value Theorem. But this is not true for R.)
   (d) Show that $M \oplus M \cong R \oplus R$ as R-modules.
   (e) Show that every finitely generated projective R-module splits as a direct sum of modules of rank 1. Deduce that $K_0(R) \cong \mathbb{Z} \oplus \mathbb{Z}/2$. (The torsion subgroup is generated by $[M] - [R]$.)

2. Let $R = \mathbb{Z}[\sqrt{-5}]$. You may assume the result of part (1) of Exercise 1.4.20 in the text, that R is a Dedekind domain. Show that $\mathfrak{p} = (2, 1 + \sqrt{-5})$ is a prime ideal which is not principal but that $\mathfrak{p}^2 = (2)$ is principal, and thus deduce that $\mathfrak{p}$ defines an element of order 2 in the class group of R. (Compare parts (2)–(4) of Exercise 1.4.20 in the text.)

3. (Compare with Exercise 1.4.21 in the text.) Let $R = \mathbb{Z}[\sqrt{5}]$, which is an integral domain with field of fractions $F = \mathbb{Q}(\sqrt{5})$. Show that $J = (1 + \sqrt{5}, 2)$ is a non-zero ideal of R with no inverse (as a

fractional ideal), and thus that R is *not* a Dedekind domain. (However, R embeds in the Dedekind domain $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$.) Hint: If J is invertible, then necessarily

$$J^{-1} = \{a + b\sqrt{5} : a, b \in \mathbb{Q}, \quad 2(a + b\sqrt{5}) \in R, \quad (a + b\sqrt{5})(1 + \sqrt{5}) \in R\}.$$

Examine these conditions, show that they define a fractional ideal I, but that $IJ \subsetneq R$.