MATH 608K (Algebraic K-Theory)
Homework Assignment #2, Fall, 2005

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due November 4, 2005

1. Do Problem 2.5.19 in the text. In other words, show that K_1 is split exact, in that if $I \triangleleft R$ and there is a splitting $R \overset{\leftarrow\cdots}{\twoheadrightarrow} R/I$, then $K_1(R)$ splits as $K_1(R/I) \oplus K_1(R, I)$.

2. Do Problem 2.5.20 in the text. In other words, show that if k is a field and R is the ring of upper-triangular matrices over k, with I the ideal of strictly upper-triangular matrices, then $K_1(R, I)$ vanishes, whereas if R' is the subring of R consisting of upper-triangular matrices with both diagonal entries equal, then I is also an ideal in R' and $K_1(R', I) \cong k$. You should use the fact that there are splittings $R \overset{\leftarrow\cdots}{\twoheadrightarrow} R/I$ and $R' \overset{\leftarrow\cdots}{\twoheadrightarrow} R'/I$, so that (by the previous exercise), $K_1(R) \cong K_1(R/I) \oplus K_1(R, I)$ and $K_1(R') \cong K_1(R'/I) \oplus K_1(R', I)$, with $R/I \cong k \times k$ and $R'/I \cong k$. On the other hand, show that $R' \cong k[t]/(t^2)$ is local, so $K_1(R')$ is easily computable. Even if you can't compute $K_1(R)$, you should at least be able to show that every element of $K_1(R', I)$ is killed under the natural map $K_1(R', I) \to K_1(R, I)$, so this is enough to show that the analogue of the Excision Theorem fails for K_1, even though it holds for K_0.

3. Show that the Whitehead group of $G = (\mathbb{Z}/2) \times (\mathbb{Z}/2)$ vanishes, by following some of the same ideas as in the proof for $\mathbb{Z}/2$ (Theorem 2.4.3 in the text). As a hint, note that $\mathbb{Z}G \cong \mathbb{Z}[s, t]/(s^2 - 1, t^2 - 1) \hookrightarrow (\mathbb{Z})^4$ (via the four irreducible representations of G sending each of s and t to each of ± 1), and identify the image.