MATH 608K (Algebraic K-Theory)
Homework Assignment #2, Fall, 2005
Partial Solutions

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2. Do Problem 2.5.20 in the text. In other words, show that if k is a field and R is the ring of upper-triangular matrices over k, with I the ideal of strictly upper-triangular matrices, then $K_1(R, I)$ vanishes, whereas if R' is the subring of R consisting of upper-triangular matrices with both diagonal entries equal, then I is also an ideal in R' and $K_1(R', I) \cong k$. You should use the fact that there are splittings $R \twoheadrightarrow R/I$ and $R' \twoheadrightarrow R'/I$, so that (by the previous exercise), $K_1(R) \cong K_1(R/I) \oplus K_1(R, I)$ and $K_1(R') \cong K_1(R'/I) \oplus K_1(R', I)$, with $R/I \cong k \times k$ and $R'/I \cong k$. On the other hand, show that $R' \cong k[t]/(t^2)$ is local, so $K_1(R')$ is easily computable. Even if you can't compute $K_1(R)$, you should at least be able to show that every element of $K_1(R', I)$ is killed under the natural map $K_1(R', I) \to K_1(R, I)$, so this is enough to show that the analogue of the Excision Theorem fails for K_1, even though it holds for K_0.

   Solution. Most people got all of this except for the part about computation of $K_1(R)$. The ideal I is contained in the radical of R, so by the argument in the proof of Proposition 2.2.4 in the text, $R^\times$ surjects onto $K_1(R)$. And $R^\times$ is given by

$$R^\times = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b \in k^\times,\ c \in k \right\}.$$

   As long as k has more than 2 elements, $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ is a commutator in this group, since we can choose $a \in k^\times$, $a \neq 1$, and then

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & (a-1)^{-1}b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & (a-1)^{-1}b \\ 0 & 1 \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} a & a(a-1)^{-1}b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & -a^{-1}(a-1)^{-1}b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (a-1)^{-1}b(-1+a) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

   So as long as k has more than 2 elements, $R^\times_{\text{ab}} \cong k^\times \times k^\times \cong (R/I)^\times$ and (since $R \to R/I$ splits, because of problem 1) $K_1(R, I) = 1$.

   There is one exceptional case, when $k = \mathbb{F}_2$. Then $K_1(R/I) \cong k^\times \times k^\times = 1$ but $R^\times_{\text{ab}} \cong \mathbb{Z}/2$, with generator $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. However, if we look at $GL(2, R, I)$, this is the group of 2×2 matrices over R

which are congruent to the identity modulo I, i.e., of the form

$$\begin{pmatrix} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \end{pmatrix}.$$

(Henceforth we drop the inner parentheses and think of $GL(2,R,I)$ as a group of 4×4 matrices over $\mathbb{F}_2$.) So $GL(2,R,I)$ has order 16. Inside this group, g corresponds to the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = e_{12}(1)$$

in the usual notation for elementary matrices in $E(4,\mathbb{F}_2)$. In fact $GL(2,R,I)$ also contains $e_{32}(1)$, $e_{14}(1)$, and $e_{34}(1)$, as is generated by these four elementary matrices. Now $E(2,R,I)$ is by definition the smallest normal subgroup of $E(2,R)$ containing the elementary 2×2 matrices with off-diagonal elements in I, in other words, containing what we have called $e_{14}(1)$ and $e_{34}(1)$. But $E(2,R)$ contains the element corresponding to $e_{42}(1)$, and $[e_{14}(1), e_{42}(1)] = e_{12}(1)$, so $g \in [E(R), E(R,I)] = E(R,I)$ and goes to 0 in $K_1(R,I)$.

3. Show that the Whitehead group of $G = (\mathbb{Z}/2) \times (\mathbb{Z}/2)$ vanishes, by following some of the same ideas as in the proof for $\mathbb{Z}/2$ (Theorem 2.4.3 in the text). As a hint, note that $\mathbb{Z}G \cong \mathbb{Z}[s,t]/(s^2-1, t^2-1) \hookrightarrow (\mathbb{Z})^4$ (via the four irreducible representations of G sending each of s and t to each of ± 1), and identify the image.

Solution. Let χ_1 and χ_2 be the homomorphisms $G \to \{\pm 1\}$ defined by $\chi_1(s) = -1$, $\chi_1(t) = 1$, $\chi_2(s) = 1$, $\chi_2(t) = -1$. Then 1 (the trivial representation $G \to \{1\}$), χ_1, χ_2, and $\chi_1\chi_2$ all induce ring homomorphisms $\mathbb{Z}G \to \mathbb{Z}$, so $(1, \chi_1, \chi_2, \chi_1\chi_2)$ gives a ring homomorphism $\varphi\colon \mathbb{Z}G \to \mathbb{Z}^4$. The image is the lattice generated by $(1, \chi_1, \chi_2, \chi_1\chi_2)(1)$, $(1, \chi_1, \chi_2, \chi_1\chi_2)(s)$, $(1, \chi_1, \chi_2, \chi_1\chi_2)(t)$, and $(1, \chi_1, \chi_2, \chi_1\chi_2)(st)$, or by $(1,1,1,1)$, $(1,-1,1,-1)$, $(1,1,-1,-1)$, and $(1,-1,-1,1)$. Since

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} = (\text{add rows 2, 3 , 4 to first row})$$

$$= \det \begin{pmatrix} 4 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} = 4 \det \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix} = 16 \neq 0,$$

the homomorphism φ is injective and its image is a lattice of index 16 in $\mathbb{Z}$; in fact, it's clear that $\operatorname{im} \varphi$ is the subring Λ of $\mathbb{Z}$ defined by

$$\Lambda = \{(a,b,c,d) \in \mathbb{Z}^4 : a \equiv b \equiv c \equiv d \mod 2,\ a+b+c+d \equiv 0 \mod 4\}.$$

We have

$$\Lambda^\times = \{(a,b,c,d) \in \mathbb{Z}^4 : a,b,c,d = \pm 1;\ 4 \mid (a+b+c+d)\} \cong \{\pm 1\} \times G,$$

so we need to show that $SK_1(\Lambda) = 1$.

Now any element of $SL(n, \Lambda)$ can be viewed as a 4-tuple (A, B, C, D) with $A, B, C, D \in SL(n, \mathbb{Z})$, $A \equiv B \equiv C \equiv D \equiv 0 \mod 2$, $A + B + C + D \equiv 0 \mod 4$. We need to show such a 4-tuple can be reduced to 1 using elementary operations over Λ.

As many members of the class realized, $(A^{-1}, A^{-1}, A^{-1}, A^{-1})$ lies in $E(n, \Lambda)$, so multiplying by this, we can assume $A = 1$. Then $(1, B^{-1}, B^{-1}, 1)$ also lies in $E(n, \Lambda)$ (the congruence conditions come from the fact that $B \equiv 1 \mod 2$, hence $B^{-1} \equiv 1 \mod 2$ and $2 + 2B^{-1} \equiv 2 + 2 \equiv 0 \mod 4$), so multiplying by this, we can assume $B = 1$ also. Similarly, we can reduce to the case $C = 1$. So we come down to the case of a 4-tuple of matrices of the form $(1, 1, 1, D)$, where $3 + D \equiv 0 \mod 4$, or in other words, $D \equiv 1 \mod 4$. So the problem reduces to showing that $SK_1(\mathbb{Z}, (4))$ is trivial.

There are several ways of proving this, but the following is probably easiest. As in the book (Theorem 2.5.12) one can reduce to considering "relative Mennicke symbols" $[a,\, b]_{(4)}$, where $a, b \in \mathbb{Z}$ and $a \equiv 1$, $b \equiv 0 \mod 4$. One has the relations $[a,\, b]_{(4)} = [a + bc,\, b]_{(4)} = [a,\, b + 4ca]_{(4)}$ for $c \in Z$. Let $b = 4k$; then we can write these as

$$[a,\, 4k]_{(4)} = [a + 4kc,\, 4k]_{(4)} = [a,\, 4(k + ca)]_{(4)}.$$

If $a = 1$, the symbol $[a,\, 4k]_{(4)}$ is trivial, and we need to show we can reduce to this case.

If $4|k| < |a|$, then writing $a = 4kq + r$ with $|r| < 4|k|$ and using the first relation above to replace a by $a - 4kq$, we can reduce the magnitude of a. In fact, we can come down to the case where $|a| < 2|k|$, since if $2|k| < |a| < 4|k|$, we can replace a by $a \pm 4k$, with the sign chosen to make this smaller than a in absolute value. Once we've reduced to the case $|a| < 2|k|$, we can replace $4k$ by $4(k \pm a)$ and reduce the magnitude of k. Iterating these processes, we can continue reducing our symbol until $a = 1$.