MATH 608K (Algebraic K-Theory)
Homework Assignment #4, Fall, 2005
K_2 and Symbols

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due Wednesday, December 14, 2005

1. Do Exercise 4.4.28 in the book. In other words, show the following about the quaternion algebras $A_F(a, b)$ (with basis 1, x, y, xy, and relations $x^2 = a \in F^\times$, $y^2 = b \in F^\times$, and $xy = -yx$) over a field F of characteristic $\neq 2$:

   (a) Show that $A_F(a, b) \cong A_F(b, a)$ is anti-isomorphic to itself, and thus defines an element of order 2 in the Brauer group of F. (What this means in concrete terms is that $A_F(a, b) \otimes_F A_F(a, b) \cong M_4(F)$.)

   (b) Show that $A_F(a, b) \otimes_F A_F(a, c) \cong M_2(A_F(a, bc))$, for $a, b, c \in F^\times$, that $A_F(a, -a) \cong M_2(F)$, and that $A_F(a, 1 - a) \cong M_2(F)$ for $a \neq 1$.

   (c) Show in this way that one gets a homomorphism $\{a, b\} \mapsto [A_F(a, b)]$ from $K_2(F)$ to a 2-torsion subgroup of the Brauer group of F, generated by stable isomorphism classes of quaternion algebras over F.

2. See if you can give an explicit calculation of $K_2(\mathbb{Q}(i))$, following the same outline as for $K_2(\mathbb{Q})$, except that remember you want to replace ordinary primes by primes in the Euclidean ring of Gaussian integers. These primes, modulo multiplication by units, are: $1 + i$, which up to a unit is the same as $1 - i$, $a \pm bi$ with $a > b > 0$, $a^2 + b^2 = p$, when p is a prime $\equiv 1\,(4)$, and ordinary primes $p \equiv 3\,(4)$.